

Odds and Ends on Finite Group Actions and Traces

Luc Illusie* and Weizhe Zheng†

Abstract

In this article, we study several problems related to virtual traces for finite group actions on schemes of finite type over an algebraically closed field. We also discuss applications to fixed point sets. Our results generalize previous results obtained by Deligne, Laumon, Serre and others.

Summary

0	Introduction	1
1	An equivariant form of a theorem of Laumon on Euler-Poincaré characteristics	3
2	A generalization to Deligne-Mumford stacks	8
3	Free actions and vanishing theorems: ℓ-adic and Betti cohomologies	18
4	Tameness at infinity	24
5	The case of rigid cohomology	29
6	Fixed point sets: around a theorem of P. Smith	33
7	A localization theorem	38

0 Introduction

Let k be an algebraically closed field of characteristic p , and let X be a k -scheme, separated and of finite type, endowed with an action of a finite group G . If ℓ is a prime number $\neq p$, G acts on $H^*(X, \mathbb{Q}_\ell)$ (resp. $H_c^*(X, \mathbb{Q}_\ell)$), and, for $s \in G$, we can consider the virtual traces

$$(0.1) \quad t_\ell(s) := \sum (-1)^i \operatorname{Tr}(s, H^i(X, \mathbb{Q}_\ell)),$$

$$(0.2) \quad t_{c,\ell}(s) := \sum (-1)^i \operatorname{Tr}(s, H_c^i(X, \mathbb{Q}_\ell)).$$

These are ℓ -adic integers. Several natural questions arise:

*Université de Paris-Sud, Département de mathématique, Bât. 425, 91405 Orsay Cedex, France; Luc.Illusie@math.u-psud.fr

†Columbia University, Department of Mathematics, 2990 Broadway, New York, NY 10027, USA; zheng@math.columbia.edu

- (a) Is $t_\ell(s)$ (resp. $t_{c,\ell}(s)$) an integer independent of ℓ ?
- (b) Do we have $t_\ell(s) = t_{c,\ell}(s)$?
- (c) Under suitable assumptions on the action of G (freeness, tameness), can one describe the virtual representation

$$(0.3) \quad \chi(X, G, \mathbb{Q}_\ell) = \sum (-1)^i [H^i(X, \mathbb{Q}_\ell)]$$

(and its analogue with compact supports) in the Grothendieck group $R_{\mathbb{Q}_\ell}(G)$ of finite dimensional \mathbb{Q}_ℓ -representations of G , where $[-]$ denotes a class in $R_{\mathbb{Q}_\ell}(G)$?

- (d) How do the numbers $t_\ell(s)$ (resp. $t_{c,\ell}(s)$) compare with the similar ones defined using other cohomology theories (rigid, for example, if $p > 1$, or Betti, when $k = \mathbb{C}$)?

These are old questions, and for some of them, partial answers were obtained long ago. Recent work of Serre ([Ser07], [Ser09], [Ser10]) has revived interest in them. The purpose of this paper is to collect answers and discuss some applications.

For $s = 1$, $t_\ell(s)$ (resp. $t_{c,\ell}(s)$) is the Euler-Poincaré characteristic $\chi(X, \mathbb{Q}_\ell)$ (resp. $\chi_c(X, \mathbb{Q}_\ell)$). By Grothendieck's trace formula, $\chi_c(X, \mathbb{Q}_\ell)$ is an integer independent of ℓ , and it is known, by a theorem of Laumon [Lau81], that it is equal to $\chi(X, \mathbb{Q}_\ell)$. In § 1 we show that, for all $s \in G$, $t_\ell(s) = t_{c,\ell}(s)$. Actually, we establish a relative, equivariant form of Laumon's theorem. In § 2 we generalize this to Deligne-Mumford stacks of finite type over a regular base of dimension at most 1 under an additional hypothesis. We also give an analogue for torsion coefficients.

By a theorem of Deligne-Lusztig [DL76, 3.3], $t_{c,\ell}(s)$ is an integer $t(s)$ independent of ℓ . In § 3 we deduce vanishing theorems for $t(s)$, $s \neq 1$, and prove a generalization (3.8) of a divisibility theorem of Serre [Ill06, 7.5].

The vanishing theorem 3.3 for free actions was shown by Deligne [Ill06] to hold more generally under a certain tameness assumption. In § 4 we consider actions of G on X that are not necessarily free. Using Vidal's groups $K(Y)_t^0$ ([Vid04], [Vid05]) we define a notion of *virtual tameness* for the action of G , and establish in this case a formula (4.11) for $\chi(X, G, \mathbb{Q}_\ell)$ (0.3) as a sum of certain induced characters. This is an algebraic analogue of a formula of Verdier [Ver73].

In § 5 we consider Berthelot's rigid cohomology with compact supports $H_{c,\text{rig}}^*(X/K)$ (where K is the fraction field of $W(k)$) and, for $s \in G$, the corresponding virtual traces

$$t_{c,\text{rig}}(s) = \sum (-1)^i \text{Tr}(s, H_{c,\text{rig}}^i(X/K)).$$

We show that $t_{c,\text{rig}}(s) = t(s)$. The proof uses de Jong's alterations to reduce to the case where X/k is projective and smooth, in which case this equality was known.

A corollary of the vanishing theorems of § 3 is that, if G is an ℓ -group, then $\chi(X)$ is congruent to $\chi(X^G)$ modulo ℓ (6.1), where X^G denotes the fixed point scheme. When X is mod ℓ acyclic, i.e. $H^*(X, \mathbb{F}_\ell) = H^0(X, \mathbb{F}_\ell) = \mathbb{F}_\ell$, one can say more: X^G is also mod ℓ acyclic. This is an analogue of a well known theorem of P. Smith [Smi38]. Here ℓ may be equal to p . This analogue is established by Serre [Ser09, 7.5]. In § 6 we give a different proof, based on equivariant cohomology $H_G^*(X, \mathbb{F}_\ell)$, for G cyclic of order ℓ , in the spirit of Borel [Bor55]. However, in contrast with the method in [Bor55], we give a shortcut, exploiting the graded module structure of $H_G^*(X, \mathbb{F}_\ell)$ over the graded algebra $H^*(G, \mathbb{F}_\ell)$. The key point is

that the restriction homomorphism $H_G^*(X, \mathbb{F}_\ell) \rightarrow H_G^*(X^G, \mathbb{F}_\ell)$ is injective and its cokernel has bounded degree. This was inspired by *localization theorems* of Quillen [Qui71, 4.2, 4.4] and of Borel-Atiyah-Segal, cf. [Hsi75, III § 2], [GKM98, § 6]. We also prove, along the same lines, that if X is a mod ℓ cohomology sphere, then so is X^G (6.11). Finally, in § 7 we prove an analogue of one of Quillen's theorems [Qui71, 4.2].

In a future paper [IZ11], we will prove analogues of other main theorems of Quillen [Qui71, 2.1, 6.2] on the structure of the mod ℓ equivariant cohomology ring.

1 An equivariant form of a theorem of Laumon on Euler-Poincaré characteristics

1.1. Fix a field k of characteristic p , an algebraic closure \bar{k} of k , a prime number $\ell \neq p$, an algebraic closure $\overline{\mathbb{Q}_\ell}$ of \mathbb{Q}_ℓ . Let X be a k -scheme, separated and of finite type. If X is smooth, Poincaré duality implies

$$(1.1.1) \quad \chi(X, \mathbb{Q}_\ell) = \chi_c(X, \mathbb{Q}_\ell).$$

It is known, by a theorem of Laumon [Lau81], that (1.1.1) holds more generally without the smoothness assumption on X . Laumon also established a relative version of this result [Lau81, 1.1]. We refer the reader to the abstract of [Lau81] for a history of Laumon's theorem. In this section, we generalize the relative form of Laumon's theorem to the equivariant situation.

1.2. Let X be a k -scheme, separated and of finite type, endowed with an action of a finite group G (G acting trivially on $\text{Spec } k$). In the sequel, unless otherwise stated, groups are supposed to act on the right. We denote by $D_c^b(X, G, \overline{\mathbb{Q}_\ell})$ the category of G -equivariant ℓ -adic complexes defined in [Zhe09, 1.3], and $K(X, G, \overline{\mathbb{Q}_\ell})$ the corresponding Grothendieck group. For $L \in D_c^b(X, G, \overline{\mathbb{Q}_\ell})$, we denote by $[L]$ its class in $K(X, G, \overline{\mathbb{Q}_\ell})$. Following Laumon [Lau81], we denote by $K^\sim(X, G, \overline{\mathbb{Q}_\ell})$ the quotient of $K(X, G, \overline{\mathbb{Q}_\ell})$ by the ideal generated by the image of $[\overline{\mathbb{Q}_\ell}(1)_{\text{Spec } k}] - 1$, and by x^\sim the image in $K^\sim(X, G, \overline{\mathbb{Q}_\ell})$ of an element x of $K(X, G, \overline{\mathbb{Q}_\ell})$.

Passing from K to K^\sim destroys a lot of arithmetic information. For example, the function $x \mapsto (\text{Tr}(-, x_{\bar{k}}), (s, g) \in G \times \text{Gal}(\bar{k}/k) \mapsto \text{Tr}(sg, x_{\bar{k}}))$ from $K(\text{Spec}(k), G, \overline{\mathbb{Q}_\ell})$ to the set of continuous functions from $G \times \text{Gal}(\bar{k}/k)$ to $\overline{\mathbb{Q}_\ell}$ does not pass to the quotient, as the case where $G = \{1\}$ and $k = \mathbb{F}_q$ trivially shows. However, not all is lost. For example, if k is a local field (fraction field of an excellent henselian discrete valuation ring), with residue field k_0 , then the restrictions of these trace functions to $G \times I$, where I is the inertia group, pass to the quotient.

Recall [Zhe09, 1.5] that, for an equivariant map $(f, u): (X, G) \rightarrow (Y, H)$, we have exact functors

$$Rf_*: D_c^b(X, G, \overline{\mathbb{Q}_\ell}) \rightarrow D_c^b(Y, H, \overline{\mathbb{Q}_\ell}), \quad Rf_!: D_c^b(X, G, \overline{\mathbb{Q}_\ell}) \rightarrow D_c^b(Y, H, \overline{\mathbb{Q}_\ell}),$$

inducing homomorphisms

$$f_*: K(X, G, \overline{\mathbb{Q}_\ell}) \rightarrow K(Y, H, \overline{\mathbb{Q}_\ell}), \quad f_!: K(X, G, \overline{\mathbb{Q}_\ell}) \rightarrow K(Y, H, \overline{\mathbb{Q}_\ell}),$$

and

$$f_*: K^\sim(X, G, \overline{\mathbb{Q}_\ell}) \rightarrow K^\sim(Y, H, \overline{\mathbb{Q}_\ell}), \quad f_!: K^\sim(X, G, \overline{\mathbb{Q}_\ell}) \rightarrow K^\sim(Y, H, \overline{\mathbb{Q}_\ell})$$

by passing to the quotients.

The following is a generalization of Laumon's theorem [Lau81, 1.1]:

Theorem 1.3. *Let $(f, u): (X, G) \rightarrow (Y, H)$ be an equivariant map between k -schemes separated and of finite type, endowed with finite group actions. Then, for any $x \in K(X, G, \overline{\mathbb{Q}_\ell})$, we have*

$$f_*(x^\sim) = f_!(x^\sim)$$

in $K^\sim(Y, H, \overline{\mathbb{Q}_\ell})$.

In particular, taking $Y = \operatorname{Spec} k$, $f: X \rightarrow Y$ the structural morphism, $H = G$, $u = \operatorname{Id}$, we get:

Corollary 1.4. *Assume k algebraically closed. Let $\chi_c(X, G, \mathbb{Q}_\ell)$ (resp. $\chi_c(X, G, \mathbb{Q}_\ell)$) be the image of $R\Gamma_c(X, \mathbb{Z}_\ell)$ (resp. $R\Gamma(X, \mathbb{Z}_\ell)$) in the Grothendieck group $R_{\mathbb{Q}_\ell}(G) = K(\operatorname{Spec} k, G, \mathbb{Q}_\ell)$ of finite dimensional \mathbb{Q}_ℓ -representations of G , i.e.*

$$\chi_c(X, G, \mathbb{Q}_\ell) = \sum (-1)^i [H_c^i(X, \mathbb{Q}_\ell)]$$

(resp.

$$\chi(X, G, \mathbb{Q}_\ell) = \sum (-1)^i [H^i(X, \mathbb{Q}_\ell)]).$$

Then we have

$$(1.4.1) \quad \chi_c(X, G, \mathbb{Q}_\ell) = \chi(X, G, \mathbb{Q}_\ell).$$

In other words, with the notations of (0.1) and (0.2), for $s \in G$, we have:

$$(1.4.2) \quad t_\ell(s) = t_{c,\ell}(s).$$

For $Y = \operatorname{Spec} k$ and $H = \{1\}$, we get:

Corollary 1.5. *Assume k algebraically closed. Set*

$$\begin{aligned} \chi_G(X, \mathbb{Q}_\ell) &= \sum (-1)^i \dim H_G^i(X, \mathbb{Q}_\ell), \\ \chi_{c,G}(X, \mathbb{Q}_\ell) &= \sum (-1)^i \dim H_{c,G}^i(X, \mathbb{Q}_\ell), \end{aligned}$$

where $H_G^i(X, \mathbb{Q}_\ell) = R^i f_* \mathbb{Q}_\ell$ (resp. $H_{c,G}^i(X, \mathbb{Q}_\ell) = R^i f_! \mathbb{Q}_\ell$) is the equivariant cohomology of X/k with no supports (resp. with compact supports). Then we have:

$$(1.5.1) \quad \chi_{c,G}(X, \mathbb{Q}_\ell) = \chi_G(X, \mathbb{Q}_\ell).$$

By definition, the number in (1.5.1) is the Euler-Poincaré characteristic of the Deligne-Mumford stack $[X/G]$.

Remark 1.6. (a) If X is smooth, Poincaré duality implies $t_\ell(s) = t_{c,\ell}(s^{-1})$. Thus, in this case, (1.4.2) follows from the fact that $t_{c,\ell}(s)$ is an integer, which is a result of Deligne and Lusztig [DL76, 3.3]. See 3.2.

(b) Recall that the action of G on X is *admissible* if X is a union of G -stable open affine subschemes (cf. [SGA 1, V 1]), which implies that X/G exists as a scheme, is separated and of finite type, and the projection $\pi: X \rightarrow X/G$ is finite. In this case, (1.5.1) brings no new information as it boils down to the original form of Laumon's theorem. Indeed, one has

$$\chi_c([X/G], \overline{\mathbb{Q}_\ell}) = \chi_c(X/G, \overline{\mathbb{Q}_\ell})$$

and similarly with χ (cf. [Zhe09, 1.7 (a)]).

Finally, here is an application to local fields, coming from [Vid04, 4.2]:

Corollary 1.7. *Let K be the fraction field of an excellent henselian discrete valuation ring of residue field k , \overline{K} an algebraic closure of K . Let η_1 be a finite, normal extension of $\eta = \text{Spec } K$, with $\kappa(\eta_1)$ contained in \overline{K} , and X/η_1 a scheme separated and of finite type. We assume that a finite group G acts on $X \rightarrow \eta_1$ by η -automorphisms. Let I be the inertia group of η . Then, for $(s, g) \in G \times I$ such that s and g induce the same automorphism of η_1/η ,*

$$(1.7.1) \quad \text{Tr}((s, g), R\Gamma(X_{\overline{K}}, \mathbb{Q}_\ell)) = \text{Tr}((s, g), R\Gamma_c(X_{\overline{K}}, \mathbb{Q}_\ell)),$$

and this ℓ -adic number is an integer independent of ℓ .

By base change to the maximal unramified extension of K , we may assume the residue field separably closed. We apply 1.2 to the projection $f: X \rightarrow \eta_1$ and $1 \in K(X, G, \mathbb{Q}_\ell)$, observing that the group $K(\eta_1, G, \mathbb{Q}_\ell) (= K^\sim(\eta_1, G, \mathbb{Q}_\ell))$ can be identified with the Grothendieck group $R_{\mathbb{Q}_\ell}(\Gamma)$ of continuous \mathbb{Q}_ℓ -linear representations of $\Gamma = G \times_{\text{Aut}(\eta_1/\eta)} \text{Gal}(\overline{K}/K)$. We get (1.7.1). The last assertion is proven in [Vid04, 4.2] when the action of G on X is admissible. The general case follows by induction since X has an affine G -stable dense open subset. When the residue field is finite, it also follows from [Zhe09, 1.16].

Proof of 1.3. Factorizing (f, u) into

$$(X, G) \xrightarrow{(f, \text{Id}_G)} (Y, G) \xrightarrow{(\text{Id}_Y, u)} (Y, H)$$

and applying the formula $R(\text{Id}_Y, u)_* \simeq R(\text{Id}_Y, u)_!$ [Zhe09, 1.5 (ii), (iii)], we may assume $G = H$, $u = \text{Id}$. By Nagata's compactification theorem ([Con07], [Lüt93]) and de Jong's construction in [dJ96, 7.6], we can find (cf. [Zhe09, 3.7]) a G -equivariant compactification $f = gj$, where $g: Z \rightarrow Y$ is proper and $j: X \rightarrow Z$ is a dense open immersion. Let $i: Z - X \rightarrow Z$ be a complementary closed immersion. As in Laumon's proof, we are then reduced to showing that for any $x \in K(X, G, \overline{\mathbb{Q}_\ell})$, the image of i^*j_*x in $K^\sim(Z - X, G, \overline{\mathbb{Q}_\ell})$ is zero. Therefore, changing notations, to prove 1.2 it suffices to establish the following (equivalent) result: \square

Proposition 1.8. *Let X be a k -scheme separated and of finite type, endowed with an action of the finite group G , $i: Y \rightarrow X$ an equivariant closed immersion, $j: U = X - Y \rightarrow X$ the (equivariant) complementary open immersion. Then, for any $x \in K(U, G, \overline{\mathbb{Q}_\ell})$, we have*

$$i^*j_*(x^\sim) = 0$$

in $K^\sim(Y, G, \overline{\mathbb{Q}_\ell})$.

To prove 1.8, we start by imitating Laumon's reduction to [Lau81, 2.2.1]. Let $f: X' \rightarrow X$ be the blow-up of Y and consider the following commutative diagram with Cartesian square

$$\begin{array}{ccccc} & & X' & \xleftarrow{i'} & Y' \\ & \nearrow j' & \downarrow f & & \downarrow g \\ U & \xrightarrow{j} & X & \xleftarrow{i} & Y \end{array}$$

By proper base change,

$$i^*j_*x = i^*f_*j'_*x = g_*i'^*j'_*x.$$

We are thus reduced to the case where Y is a Cartier divisor. By cohomological descent for a finite covering of X by G -stable open subsets, we may further assume that Y is defined by a global equation $F \in \Gamma(X, \mathcal{O}_X)$. Up to replacing F by $\prod_{s \in G} s^* F$ (and changing the scheme structure of Y), we may assume that F is invariant under G . Then F defines a G -equivariant morphism $f: X \rightarrow \mathbb{A}_k^1$, G acting trivially on \mathbb{A}_k^1 , with fiber Y at $\{0\}$. As in (*loc. cit.*), replacing \mathbb{A}_k^1 by its henselization S at 0, with closed point s and generic point η , we are reduced to showing that, for any $K \in D_c^b(X_\eta, G, \overline{\mathbb{Q}_\ell})$, the class in $K^\sim(Y, G, \overline{\mathbb{Q}_\ell})$ of

$$R\Gamma(I, R\Psi K) \in D_c^b(Y, G, \overline{\mathbb{Q}_\ell})$$

is zero, where I is the inertia subgroup of $\text{Gal}(\overline{\eta}/\eta)$, $\overline{\eta}$ a separable closure of η . It then suffices to invoke the following analogue of [Lau81, 2.2.1].

Theorem 1.9. *Let S be the spectrum of a henselian discrete valuation ring, with closed point $s = \text{Spec } k$ and generic point η , Y be a scheme of finite type over s , endowed with an action of a finite group G (G acting trivially on s). Then for any $L \in D_c^b(Y \times_s \eta, G, \overline{\mathbb{Q}_\ell})$, the class in $K^\sim(Y, G, \overline{\mathbb{Q}_\ell})$ of*

$$R\Gamma(I, L) \in D_c^b(Y, G, \overline{\mathbb{Q}_\ell})$$

is zero, where I is the inertia subgroup of $\text{Gal}(\overline{\eta}/\eta)$, $\overline{\eta}$ a separable closure of η .

We refer to [Zhe09, 4.1] for the definition of $D_c^b(Y \times_s \eta, G, \overline{\mathbb{Q}_\ell})$. In our case, it is based on the topos $(Y \times_s \eta, G)^\sim$ consisting of sheaves on $Y_{\overline{s}}$ endowed with an action of $\text{Gal}(\overline{\eta}/\eta) \times G$ compatible with the action of $\text{Gal}(\overline{s}/s)$ on $Y_{\overline{s}/s}$. Here \overline{s} is a separable closure of s .

Proof. Let P_ℓ be the kernel of the ℓ -component $t_\ell: I \rightarrow I_\ell = \mathbb{Z}_\ell(1)$ of the tame character, and let $\eta_\ell = \overline{\eta}/P_\ell$. We consider the topos of G - $\text{Gal}(\eta_\ell/\eta)$ -sheaves on $Y_{\overline{s}}$, which is the subcategory of $(Y \times_s \eta, G)^\sim$ consisting of sheaves on which P_ℓ acts trivially. Since

$$R\Gamma(I, L) = R\Gamma(I_\ell, L^{P_\ell}),$$

one is reduced to showing that, for any G - $\text{Gal}(\eta_\ell/\eta)$ - $\overline{\mathbb{Q}_\ell}$ -sheaf L on $Y_{\overline{s}}$, the class of $R\Gamma(I_\ell, L)$ in $K^\sim(Y, G, \overline{\mathbb{Q}_\ell})$ is zero, where $\eta_\ell = \overline{\eta}/P_\ell$. We use the argument of Deligne at the end of [Lau81]. If σ is a topological generator of I_ℓ , we have a decomposition

$$(1.9.1) \quad L = \bigoplus_{\alpha \in \overline{\mathbb{Q}_\ell}^\times} \bigcup_{n \geq 1} \text{Ker}((\sigma - \alpha)^n, L).$$

Let L^u be the largest subsheaf of L (in the category of G - $\text{Gal}(\eta_\ell/\eta)$ - $\overline{\mathbb{Q}_\ell}$ -sheaves on $Y_{\overline{s}}$) on which the action of I_ℓ is unipotent. In terms of the above decomposition,

$$(1.9.2) \quad L^u = \bigcup_{n \geq 1} \text{Ker}((\sigma - 1)^n, L).$$

As the formation of L^u and of $R\Gamma(I_\ell, -)$ commutes with taking stalks at geometric points of Y , the inclusion $L^u \rightarrow L$ induces an isomorphism

$$R\Gamma(I_\ell, L^u) \xrightarrow{\sim} R\Gamma(I_\ell, L).$$

Therefore we may assume that the action of I_ℓ on L is unipotent. Thus there exists a (twisted) nilpotent endomorphism $N: L \rightarrow L(-1)$ (a morphism of G - $\text{Gal}(\eta_\ell/\eta)$ - $\overline{\mathbb{Q}_\ell}$ -sheaves on $Y_{\overline{s}}$) such

that the representation $\rho: I_\ell \rightarrow \text{Aut}(L)$ is given by $\rho(g) = \exp(Nt_\ell(g))$ for $g \in I_\ell$. By definition, $H^0(I_\ell, L) = L^{I_\ell}$, and we have a canonical isomorphism

$$H^1(I_\ell, L) \xrightarrow{\sim} (L_{I_\ell})(-1),$$

where L_{I_ℓ} is the sheaf of co-invariants of I_ℓ in L (given by $z \mapsto [z(\sigma)] \otimes \sigma^\vee$ on 1-cocycles of the standard cochain complex, where σ is a generator of $I_\ell = \mathbb{Z}_\ell(1)$, $\sigma^\vee \in \mathbb{Z}_\ell(-1)$ its dual, and $[-]$ means a class in L_{I_ℓ}). In other words,

$$H^0(I_\ell, L) \simeq \text{Ker } N, \quad H^1(I_\ell, L) \simeq \text{Coker } N.$$

Moreover, $H^i(I_\ell, L) = 0$ for $i \neq 0, 1$. Using the monodromy filtration [Del80, 1.6.14] $\cdots \subset M_i L \subset M_{i+1} L \subset \cdots$ of L in the category of $G\text{-Gal}(\eta_\ell/\eta)\text{-}\overline{\mathbb{Q}_\ell}$ -sheaves on $Y_{\overline{s}}$, one gets the isomorphisms

$$\text{Gr}_i^M((\text{Coker } N)(1))(i) \xrightarrow{\sim} \text{Gr}_{-i}^M(\text{Ker } N),$$

which implies that $\text{Ker } N$ and $\text{Coker } N$ have the same image in $K^\sim(I_\ell, \overline{\mathbb{Q}_\ell})$. \square

The analogue of 1.9 (and, in turn, 1.8 and 1.3) with $\overline{\mathbb{Q}_\ell}$ replaced by an algebraic extension E of a complete discrete valuation field of characteristic $(0, \ell)$ still holds. In fact, in this case, although we may no longer have a decomposition of L as in (1.9.1), the expression for L^u still holds.

1.10. With the notations and hypotheses of 1.8, assume k algebraically closed and X proper. Let $L \in D_c^b(U, G, \overline{\mathbb{Q}_\ell})$, and $s \in G$ such that the fixed point set X^s of s is contained in Y . By 1.3 we know that

$$(1.10.1) \quad \text{Tr}(s, R\Gamma_c(U, L)) = \text{Tr}(s, R\Gamma(U, L)).$$

Though s has no fixed points on U , this trace can be nonzero (for example, if X is the affine line over k , with $p > 1$, and G the cyclic group $\mathbb{Z}/p\mathbb{Z}$ acting on X by translation, with generator $s: x \mapsto x + 1$, then $t_c(s) = 1$). We can rewrite both sides as

$$\begin{aligned} \text{Tr}(s, R\Gamma_c(U, L)) &= \text{Tr}(s, R\Gamma(X, j_! L)), \\ \text{Tr}(s, R\Gamma(U, L)) &= \text{Tr}(s, R\Gamma(X, Rj_* L)). \end{aligned}$$

By the Lefschetz-Verdier trace formula [SGA 5, III 4.7], each of these traces is a sum of “local terms at infinity”, associated with the connected components of $X^s \subset Y$:

$$\begin{aligned} \text{Tr}(s, R\Gamma(X, j_! L)) &= \sum_{Z \in \pi_0(X^s)} \text{Tr}(s, R\Gamma(X, j_! L))_Z, \\ \text{Tr}(s, R\Gamma(X, Rj_* L)) &= \sum_{Z \in \pi_0(X^s)} \text{Tr}(s, R\Gamma(X, Rj_* L))_Z, \end{aligned}$$

where the subscript Z means the local Verdier term at Z for the correspondence defined by $s: X \rightarrow X$ and $s^*: j_! L \rightarrow j_! L$ or $s^*: Rj_* L \rightarrow Rj_* L$. We have the following refinement of (1.10.1):

Corollary 1.11. *With the notations and hypotheses of 1.10, for each $Z \in \pi_0(X^s)$, we have*

$$\text{Tr}(s, R\Gamma(X, j_! L))_Z = \text{Tr}(s, R\Gamma(X, Rj_* L))_Z.$$

By the additivity of $\mathrm{Tr}(s, -)_Z$, we have

$$\mathrm{Tr}(s, R\Gamma(X, j_!L))_Z - \mathrm{Tr}(s, R\Gamma(X, Rj_*L))_Z = \mathrm{Tr}(s, R\Gamma(X, i_*i^*Rj_*L))_Z.$$

By the (trivial) Lefschetz-Verdier formula for $i: Y \rightarrow X$ [SGA 5, III 4.4],

$$\mathrm{Tr}(s, R\Gamma(X, i_*i^*Rj_*L))_Z = \mathrm{Tr}(s, R\Gamma(Y, i^*Rj_*L))_Z.$$

By definition [SGA 5, III 4.7], if Id_E denotes the identity correspondence on $E = i^*Rj_*L$,

$$\mathrm{Tr}(s, R\Gamma(Y, i^*Rj_*L))_Z = (a_Z)_* \langle s, \mathrm{Id}_E \rangle_Z,$$

where $a_Z: Z \rightarrow \mathrm{Spec} k$ is the projection, $\langle s, \mathrm{Id}_E \rangle_Z \in H^0(Z, K_Z)$ is the Verdier term at Z for the correspondences s and Id_E [SGA 5, III (4.2.7)], $K_Z = Ra_Z^! \overline{\mathbb{Q}}_\ell$, and $(a_Z)_*: H^0(Z, K_Z) \rightarrow \overline{\mathbb{Q}}_\ell$ is the trace map, defined by the adjunction map $Ra_* K_Z \rightarrow \overline{\mathbb{Q}}_\ell$. More generally, for any $F \in D_c^b(Y, G, \overline{\mathbb{Q}}_\ell)$ and $Z \in \pi_0(X^s)$, we have a Verdier term $\langle s, \mathrm{Id}_F \rangle_Z \in H^0(Z, K_Z)$. By [SGA 5, III (4.13.1)], $\langle s, \mathrm{Id}_F \rangle_Z$ is additive in F , hence depends only on the class of F in $K(Y, G, \overline{\mathbb{Q}}_\ell)$. Therefore, by 1.8, we have $\langle s, \mathrm{Id}_E \rangle_Z = 0$, which completes the proof.

1.12. Let us mention some analogues of Laumon's theorem in topology. Let X be an n -dimensional manifold. If Poincaré duality holds for X , then we have the following analogue of 1.1:

$$\chi(X, \mathbb{Q}) = (-1)^n \chi_c(X, \mathbb{Q}).$$

In particular, odd-dimensional compact manifolds have vanishing Euler-Poincaré characteristic. More generally, Sullivan [Sul71] has shown that compact stratified spaces (in the sense of Thom) with odd-dimensional strata have vanishing Euler-Poincaré characteristic. Weinberger, Goresky and MacPherson used this to show that $\chi(X, \mathbb{Q}) = \chi_c(X, \mathbb{Q})$ holds for all stratified spaces X with even-dimensional strata. See [Ful93, p. 141, Note 13].

2 A generalization to Deligne-Mumford stacks

In the situation of 1.2, we have

$$D_c^b(X, G, \overline{\mathbb{Q}}_\ell) \simeq D_c^b([X/G], \overline{\mathbb{Q}}_\ell),$$

where $[X/G]$ denotes the Deligne-Mumford stack associated to the action of G on X . Moreover, the equivariant operations Rf_* , $Rf_!$ correspond to similar operations for the associated morphisms of Deligne-Mumford stacks. In the first half of this section we show that 1.3 extends to morphisms of Deligne-Mumford stacks of finite type over a regular base of dimension ≤ 1 satisfying the condition (A) below. In the second half, we establish an analogue for torsion coefficients. The results of this section will not be used in the following ones with the exception of 3.1, where only the extension of 1.3 to algebraic spaces is used.

2.1. In this section, unless otherwise stated, we fix a (Noetherian) regular base scheme S of dimension ≤ 1 satisfying the condition

- (A) Every nonempty scheme of finite type over S has a nonempty geometrically unibranch [EGA IV, 6.15.1] open subscheme.

This condition is satisfied if S is a Nagata scheme [The, 033S] (by [EGA IV, 9.7.10]) or if S is semi-local. We fix a prime number ℓ invertible on S and an algebraic extension E of a complete discrete valuation field E_0 of characteristic $(0, \ell)$. We use the convention of [LMB00, 4.1] for Deligne-Mumford stacks. In particular, the diagonal of a Deligne-Mumford stack is assumed to be quasi-compact and separated. For a Deligne-Mumford S -stack \mathcal{X} of finite type, we denote by $D_c^b(\mathcal{X}, E)$ the category of bounded E -complexes. If S is affine and excellent and if all schemes of finite type over S has finite ℓ -cohomological dimension, the construction of $D_c^b(\mathcal{X}, E)$ and of the corresponding six operations is done in [LO08]. For the general case, see [Zhe11]. Note that the following sections do not depend on [Zhe11] because there we work over an algebraically closed field.

We denote by $K(\mathcal{X}, E)$ the Grothendieck group of $D_c^b(\mathcal{X}, E)$. For $L \in D_c^b(\mathcal{X}, E)$, we denote by $[L]$ its class in $K(\mathcal{X}, E)$. As in 1.2, we denote by $K^\sim(\mathcal{X}, E)$ the quotient of $K(\mathcal{X}, E)$ by the ideal generated by $[E(1)] - 1$, and by x^\sim the image in $K^\sim(\mathcal{X}, E)$ of an element x of $K(\mathcal{X}, E)$.

Recall that, for a morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ of Deligne-Mumford S -stacks of finite type, we have exact functors

$$Rf_*, Rf_!: D_c^b(\mathcal{X}, E) \rightarrow D_c^b(\mathcal{Y}, E)$$

inducing homomorphisms

$$f_*, f_!: K(\mathcal{X}, E) \rightarrow K(\mathcal{Y}, E)$$

and

$$f_*, f_!: K^\sim(\mathcal{X}, E) \rightarrow K^\sim(\mathcal{Y}, E)$$

by passing to quotients.

The following is a generalization of 1.3. The proof will be given in 2.6.

Theorem 2.2. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Deligne-Mumford S -stacks of finite type. Then, for any $x \in K(\mathcal{X}, E)$, we have*

$$f_*(x^\sim) = f_!(x^\sim)$$

in $K^\sim(\mathcal{Y}, E)$.

For a point ξ of \mathcal{Y} , we denote by $i_\xi: \mathcal{G}_\xi \rightarrow \mathcal{Y}$ its residue gerbe [LMB00, 11.1]. It is isomorphic to a quotient stack $[\text{Spec } K/G]$, where K is a finite type extension field of $\kappa(s)$, $s \in S$ is the image of ξ , and G is a finite group acting on K on the left leaving $\kappa(s)$ fixed. To see this, we may assume, by [LMB00, 6.1.1], that $\mathcal{Y} = [Y/H]$ is the quotient stack of an S -scheme Y of finite type by a finite group G acting on Y on the right by k -automorphisms. A representative $\text{Spec } K_1 \rightarrow \mathcal{Y}$ of ξ corresponds to an H -torsor T over K_1 together with an H -equivariant map $t: T \rightarrow Y$. Let X be the image of t and endow it with the scheme structure $X = \coprod_{y \in X} \text{Spec } \kappa(y)$. Then t factorizes into H -equivariant maps $T \rightarrow X \rightarrow Y$. Hence $\mathcal{G}_\xi \simeq [X/H] \simeq [y/G_y]$, where $y \in X$ and $G_y < H$ is the stabilizer of y . Note that \mathcal{G}_ξ is also the residue gerbe of the point $\xi \rightarrow \mathcal{Y}_s$, where $\mathcal{Y}_s = \mathcal{Y} \times_S s$.

Lemma 2.3. *The homomorphism*

$$K^\sim(\mathcal{Y}, E) \rightarrow \prod_{\xi} K^\sim(\mathcal{G}_\xi, E)$$

induced by i_ξ^ , where ξ runs over all points of \mathcal{Y} , is an injection.*

Proof. We prove 2.3 by Noetherian induction on \mathcal{Y} . The assertion being trivial for $\mathcal{Y} = \emptyset$, we assume \mathcal{Y} nonempty. Let x be an element of $K(\mathcal{Y}, E)$ such that $i_\xi^*(x^\sim) = 0$ in $K^\sim(\mathcal{G}_\xi, E)$ for any ξ . We shall show $x^\sim = 0$. As above, by [LMB00, 6.1.1], there exists a nonempty open immersion $j: [Y/H] \hookrightarrow \mathcal{Y}$ from the quotient stack of an S -scheme Y of finite type by a finite group H acting on Y on the right by S -automorphisms. By (A), shrinking Y if necessary, we may assume Y is geometrically unibranch and connected, and $j^*x = [\mathcal{F}] - [\mathcal{F}']$ for lisse E -sheaves \mathcal{F} and \mathcal{F}' on $[Y/H]$. Here by *lisse E -sheaf* we mean a sheaf of the form $\mathcal{E} \otimes_{\mathcal{O}_{E_1}} E$, where \mathcal{O}_{E_1} is the ring of integers of a finite extension E_1 of E_0 contained in E , and \mathcal{E} is a lisse \mathcal{O}_{E_1} -sheaf. Let $i: \mathcal{Y} - [Y/H] \rightarrow \mathcal{Y}$ be a complementary closed immersion. Since $x = j_! j^* x + i_* i^* x$ and $i^*(x^\sim) = 0$ by induction hypothesis, it suffices to show $j^*(x^\sim) = 0$. We may thus assume $\mathcal{Y} = [Y/H]$, Y geometrically unibranch, $x = [\mathcal{F}] - [\mathcal{F}']$ where $\mathcal{F}, \mathcal{F}'$ are lisse E -sheaves.

Let $\bar{\eta}$ is a geometric point above the generic point η of Y . The category of lisse E -sheaves on \mathcal{Y} under our convention is equivalent to the category of (finite-dimensional) E -representations of $\pi_1(\mathcal{Y}, \bar{\eta})$ (see [Noo04, § 4] for a definition of $\pi_1(\mathcal{Y}, \bar{\eta})$). Let $K_{\text{lisse}}(\mathcal{Y}, E)$ denote the corresponding Grothendieck group. The H -equivariant map $\epsilon: \eta \rightarrow Y$ induces a morphism $\iota: [\eta/H] \rightarrow \mathcal{Y}$, which is the residue gerbe of \mathcal{Y} at the point $\eta \rightarrow \mathcal{Y}$. By hypothesis, $\iota^*(x^\sim) = 0$. To see $x^\sim = 0$, it suffices to show that

$$\iota^*: K_{\text{lisse}}^\sim(\mathcal{Y}, E) \rightarrow K^\sim([\eta/H], E)$$

is an injection.

By the Jordan-Hölder theorem, $K_{\text{lisse}}(\mathcal{Y}, E)$ (resp. $K([\eta/H], E)$) is a free abelian group with the set of isomorphism classes S_1 (resp. S_2) of simple E -representations of $\pi_1(\mathcal{Y}, \bar{\eta})$ (resp. $\pi_1([\eta/H], \bar{\eta})$) as a base. Let S_i^\sim be the quotient set of S_i by the equivalence relation \sim defined by $L \sim M$ if there exists $n \in \mathbb{Z}$ such that $L \simeq M(n)$, $i = 1, 2$. Then $K_{\text{lisse}}^\sim(\mathcal{Y}, E)$ (resp. $K^\sim([\eta/H], E)$) is a free abelian group with basis S_1^\sim (resp. S_2^\sim). We have the following morphism of short exact sequences of groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(\eta, \bar{\eta}) & \longrightarrow & \pi_1([\eta/H], \bar{\eta}) & \longrightarrow & H \longrightarrow 1 \\ & & \downarrow \pi_1(\epsilon, \bar{\eta}) & & \downarrow \pi_1(\iota, \bar{\eta}) & & \parallel \\ 1 & \longrightarrow & \pi_1(Y, \bar{\eta}) & \longrightarrow & \pi_1(\mathcal{Y}, \bar{\eta}) & \longrightarrow & H \longrightarrow 1 \end{array}$$

Since Y is geometrically unibranch, the homomorphism $\pi_1(\epsilon, \bar{\eta})$ is surjective. Hence the same is true for $\pi_1(\iota, \bar{\eta})$. The latter clearly induces an injection $S_1 \rightarrow S_2$. Therefore, for all $L, M \in S_1$ satisfying $\iota^* L \simeq (\iota^* M)(n) \simeq \iota^*(M(n))$, we have $L \simeq M(n)$. In other words, ι^* gives an injection $S_1^\sim \rightarrow S_2^\sim$. This completes the proof of 2.3. \square

2.4. The preceding lemma allows us to show that the analogue of 1.3 still holds over the base scheme S (G and H acting trivially on S) and with $\overline{\mathbb{Q}_\ell}$ replaced by E . As before, we are reduced to proving the analogue of 1.8 over S . Note that the case where S is the spectrum of a discrete valuation ring is easy (as observed by Vidal [Vid05, 0.1]).

Consider first the case where $i: Y \rightarrow X$ comes from a closed immersion $T \rightarrow S$ by base change. We may assume $T = s$ is a closed point of S . Then, replacing S by its henselization $S_{(s)}$ at s , with generic point η , we are reduced to showing that, for any $K \in D_c^b(X_\eta, G, E)$, the class in $K^\sim(Y, G, E)$ of

$$R\Gamma(I, R\Psi K) \in D_c^b(Y, G, E)$$

is zero, where I is the inertia subgroup of $\text{Gal}(\bar{\eta}/\eta)$, $\bar{\eta}$ an algebraic closure of η . It then suffices to apply 1.9.

In the general case, apply the generic base change theorem [SGA 4 $\frac{1}{2}$, Th. finitude 1.9] to find a dense open subset V of S such that the formation of j_*x commutes with any base change $S' \rightarrow V$. Consider the following diagram with Cartesian squares

$$\begin{array}{ccccccc}
U_s & \xrightarrow{w_U} & U_V & \xrightarrow{v_U} & U & \xleftarrow{t_U} & U_T \\
\downarrow j_s & & \downarrow j_V & & \downarrow j & & \downarrow j_T \\
X_s & \xrightarrow{w} & X_V & \xrightarrow{v} & X & \xleftarrow{t} & X_T \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
s & \longrightarrow & V & \longrightarrow & S & \longleftarrow & T
\end{array}$$

where $T = S - V$, s is an arbitrary point of V . We have $x = v_{U!}x_V + t_{U*}x_T$, where $x_V = v_U^*x$, $x_T = t_U^*x$. Applying 1.8 to j_s , we obtain

$$w^*j_{V*}(x_{\tilde{V}}) = j_{s*}w_U^*(x_{\tilde{V}}) = j_{s!}w_U^*(x_{\tilde{V}}) = w^*j_{V!}(x_{\tilde{V}}).$$

Hence, by 2.3, $j_{V*}(x_{\tilde{V}}) = j_{V!}(x_{\tilde{V}})$. Therefore, by the special case above,

$$j_*v_{U!}(x_{\tilde{V}}) = j_*v_{U*}(x_{\tilde{V}}) = v_*j_{V*}(x_{\tilde{V}}) = v_*j_{V!}(x_{\tilde{V}}) = v_{!}j_{V!}(x_{\tilde{V}}) = j_{!}v_{U!}(x_{\tilde{V}}).$$

On the other hand, applying 1.8 to j_T , we obtain

$$j_*t_{U*}(x_{\tilde{T}}) = t_*j_{T*}(x_{\tilde{T}}) = t_*j_{T!}(x_{\tilde{T}}) = j_{!}t_{U*}(x_{\tilde{T}}).$$

Therefore $j_*(x^\sim) = j_{!}(x^\sim)$.

The following is a variant of [LMB00, 6.2, 6.3]. We use the convention of [LMB00, 4.1] for Artin stacks. In particular, the diagonal of an Artin stack is assumed to be quasi-compact and separated.

Lemma 2.5. *Let S be a quasi-separated scheme, \mathcal{X} be an Artin S -stack, K be an S -field, $x = \text{Spec } K$, H be a finite group acting on x on the right by S -automorphisms, and let $i: [x/H] \rightarrow \mathcal{X}$ be a morphism. Then there exists a 2-commutative diagram*

$$\begin{array}{ccc}
& & [X/G] \\
& \nearrow s & \downarrow \phi \\
[x/H] & \xrightarrow{i} & \mathcal{X}
\end{array}$$

where X is an affine scheme, G is a finite group acting on X on the right by S -automorphisms and ϕ is a representable smooth morphism. Moreover, if \mathcal{X} is a Deligne-Mumford S -stack, we can choose the above diagram such that ϕ is étale and that the following square is 2-Cartesian

$$\begin{array}{ccc}
[x/H] & \xrightarrow{s} & [X/G] \\
\parallel & & \downarrow \phi \\
[x/H] & \xrightarrow{i} & \mathcal{X}
\end{array}$$

Let us recall the constructions in [LMB00, 6.6]. Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a representable [LMB00, 3.9] and separated morphism of S -stacks, $d \geq 0$. We define an S -stack $\mathrm{SEC}_d(\mathcal{X}/\mathcal{Y})$ by assigning to every affine scheme U equipped with a morphism $U \rightarrow S$, the category of arrays (x_1, \dots, x_d) of disjoint sections of the algebraic U -space $X = \mathcal{X} \times_{\mathcal{Y}} U$. The S -stack $\mathrm{SEC}_d(\mathcal{X}/\mathcal{Y})$ is equipped with a natural action of the symmetric group \mathfrak{S}_d , compatible with the projection to \mathcal{Y} . Let $\mathrm{ET}_d(\mathcal{X}/\mathcal{Y})$ be the quotient stack. For a quasi-separated S -scheme V , giving a morphism $V \rightarrow \mathrm{ET}_d(\mathcal{X}/\mathcal{Y})$ is equivalent to giving a morphism $V \rightarrow \mathcal{Y}$ and giving a subscheme Z of the algebraic space $\mathcal{X} \times_{\mathcal{Y}} V$ which is finite, étale of degree d over V ([LMB00, 6.6.3 (i)]). The structural morphism $\mathrm{ET}_d(\mathcal{X}/\mathcal{Y}) \rightarrow \mathcal{Y}$ is representable and separated [LMB00, 6.6.3 (ii)].

Proof. We prove 2.5 by imitating [LMB00, 6.7]. Take a 2-commutative diagram with 2-Cartesian squares

$$\begin{array}{ccccc} T & \longrightarrow & \mathcal{B} & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \pi \\ x & \longrightarrow & [x/H] & \xrightarrow{i} & \mathcal{X} \end{array}$$

with Z an affine scheme, π smooth and T non-empty. Then T is a smooth algebraic space over x and an H -torsor over \mathcal{B} . Let L be an H -equivariant closed subscheme of T , finite étale over x . For example, we can take a closed point t of T whose residue field is a separable extension of K , and take L to be the H -orbit of t , endowed with the reduced algebraic subspace structure. Let d be the degree of L over x . We have thus an H -equivariant section of $\mathrm{ET}_d(T/x) \rightarrow x$, giving rise to a section of $\mathrm{ET}_d(\mathcal{B}/[x/H]) \rightarrow [x/H]$, hence a 2-commutative diagram

$$\begin{array}{ccccc} & & & \mathrm{ET}_d(Z/\mathcal{X}) & \\ & \nearrow i_1 & & \downarrow & \\ x & \longrightarrow & [x/H] & \xrightarrow{i} & \mathcal{X} \end{array}$$

Since $\mathrm{ET}_d(Z/\mathcal{X})$ is a Deligne-Mumford stack smooth over \mathcal{X} [LMB00, 6.6.3 (ii)], up to replacing \mathcal{X} by $\mathrm{ET}_d(Z/\mathcal{X})$, we may assume that \mathcal{X} is a Deligne-Mumford S -stack. In this case, we can take π to be étale. Then $\mathrm{SEC}_d(Z/\mathcal{X})$ is a quasi-affine scheme [LMB00, 6.6.2 (iii)] and $\mathrm{ET}_d(Z/\mathcal{X}) = [\mathrm{SEC}_d(Z/\mathcal{X})/\mathfrak{S}_d]$ is étale over \mathcal{X} . Moreover, since T is a quasi-compact étale algebraic space over x , it is finite and we can take $L = T$, so that $\mathrm{ET}_d(\mathcal{B}/[x/H]) \rightarrow [x/H]$ is an isomorphism. The point i_1 corresponds to an \mathfrak{S}_d -orbit of $\mathrm{SEC}_d(Z/\mathcal{X})$, which is contained in an \mathfrak{S}_d -equivariant affine open. Thus 2.5 holds by taking X to be the aforementioned open and G to be the group \mathfrak{S}_d . \square

2.6. Proof of 2.2. By 2.3, it is enough to show $i_\xi^* f_*(x^\sim) = i_\xi^* f_!(x^\sim)$ for all points ξ of \mathcal{Y} . By 2.5, it is then enough to show $\phi^* f_*(x^\sim) = \phi^* f_!(x^\sim)$ for every representable étale morphism $\phi: [Y/H] \rightarrow \mathcal{Y}$ where Y is an affine S -scheme of finite type and G is a finite group acting on Y . By base change by ϕ , it is thus enough to establish 2.2 in the case where $\mathcal{Y} = [Y/H]$. In particular, 2.4 implies that 2.2 holds if f is an open immersion (with no additional assumption on \mathcal{Y}).

We prove 2.2 in the case where $\mathcal{Y} = [Y/H]$ by Noetherian induction on \mathcal{X} . Let $j: [X/G] \rightarrow \mathcal{X}$ be a dominant open immersion [LMB00, 6.1.1], where X is an affine S -scheme of finite type and G is a finite group acting on X . By [Zhe09, 5.1] (which holds over general base schemes),

up to replacing (X, G) by another pair with an isomorphic quotient stack, we may assume that $fj: [X/G] \rightarrow [Y/H]$ is induced by an equivariant map $(X, G) \rightarrow (Y, H)$. Let i be a closed immersion $\mathcal{X} = [X/G] \rightarrow \mathcal{X}$. Then $x = j_!j^*x + i_*i^*x$. Since $j_*j^*(x^\sim) = j_!j^*(x^\sim)$ by the already proven case of open immersion, we have

$$f_*j_!j^*(x^\sim) = f_*j_*j^*(x^\sim) = (fj)_*j^*(x^\sim) = (fj)_!j^*(x^\sim) = f_!j_!j^*(x^\sim)$$

by 2.4 applied to fj . On the other hand,

$$f_*i_*i^*(x^\sim) = (fi)_*i^*(x^\sim) = (fi)_!i^*(x^\sim) = f_!i_*i^*(x^\sim)$$

by induction hypothesis applied to fi . Therefore $f_*(x^\sim) = f_!(x^\sim)$. \square

In the rest of this section, we consider analogues of Laumon's theorem for torsion coefficients. Let F be a field of characteristic ℓ . The following is an analogue of Theorem 1.9.

Theorem 2.7. *Let S be the spectrum of a henselian discrete valuation ring, with closed point s and generic point η , Y be a scheme of finite type over s , endowed with an action of a finite group G (G acting trivially on s). Then for any $L \in D_c^b(Y \times_s \eta, G, F)$, the class in $K^\sim(Y, G, F)$ of*

$$R\Gamma(I, L) \in D_c^b(Y, G, F)$$

is zero, where I is the inertia subgroup of $\text{Gal}(\overline{\eta}/\eta)$, $\overline{\eta}$ an algebraic closure of η .

As in 1.9, one is reduced to showing that, for any $G\text{-Gal}(\eta_\ell/\eta)\text{-}F$ -sheaf, the class of $R\Gamma(I_\ell, L)$ in $K^\sim(Y, G, F)$ is zero, where $\eta_\ell = \overline{\eta}/P_\ell$. As before, we may assume that the action of I_ℓ on L is unipotent. Fix a topological generator σ of $I_\ell = \mathbb{Z}_\ell(1)$ and define a (nilpotent) $G\text{-}I_\ell$ -equivariant operator

$$N_\sigma: L(1) \rightarrow L$$

by the formula $N_\sigma(\bar{\sigma} \otimes a) = ua$, where $\bar{\sigma} \in F(1) = F \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)$ is the image of σ , $u = \sigma - 1: L \rightarrow L$. For $\gamma \in \text{Gal}(\eta_\ell/\eta)$, we have

$$(N_\sigma\gamma - \gamma N_\sigma)(\bar{\sigma} \otimes a) = N_\sigma((\bar{\sigma})^{\chi(\gamma)} \otimes \gamma a) - \gamma(\sigma - 1)a = \overline{\chi(\gamma)}u\gamma a - (\sigma^{\chi(\gamma)} - 1)\gamma a,$$

where $\chi: \text{Gal}(\eta_\ell/\eta) \rightarrow \mathbb{Z}_\ell^\times$ is the cyclotomic character, $\overline{\chi(\gamma)} \in \mathbb{F}_\ell^\times$ is the image of $\chi(\gamma) \in \mathbb{Z}_\ell^\times$. Since

$$\sigma^{\chi(\gamma)} - 1 = (1 + u)^{\chi(\gamma)} - 1 = \overline{\chi(\gamma)}u + u^2P(u),$$

where $P(u)$ is a polynomial in u , we have

$$(N_\sigma\gamma - \gamma N_\sigma)(\bar{\sigma} \otimes a) = \overline{\chi(\gamma)}u\gamma a - [\overline{\chi(\gamma)}u\gamma a + u^2P(u)\gamma a] = -u^2P(u)\gamma a \in \text{Im } u^2 = \text{Im } N_\sigma^2.$$

It follows that $\text{Im}(N_\sigma^m\gamma - \gamma N_\sigma^m) \subset \text{Im}(N_\sigma^{m+1})$ for $m \geq 0$. Let $\dots \subset M_i L \subset M_{i+1} L \subset \dots$ be the filtration in the category of $G\text{-}F$ -sheaves on $Y_{\overline{s}}$ characterized by $N_\sigma M_i(1) \subset M_{i-2}$ and the property that N_σ^i induces an isomorphism of $G\text{-}F$ -sheaves

$$(2.7.1) \quad \text{Gr}_i^M L(i) \xrightarrow{\sim} \text{Gr}_{-i}^M L.$$

As $\cdots \subset \gamma M_i L \subset \gamma M_{i+1} L \subset \cdots$ satisfies the same condition for any $\gamma \in \text{Gal}(\eta_\ell/\eta)$, the filtration is $\text{Gal}(\eta_\ell/\eta)$ -stable. Moreover, (2.7.1) is $\text{Gal}(\eta_\ell/\eta)$ -equivariant, hence the same holds for the isomorphism

$$\text{Gr}_i^M(\text{Coker } N_\sigma)(i) \simeq \text{Gr}_{-i}^M(\text{Ker } N_\sigma(-1)).$$

Therefore $H^0(I_\ell, L) = L^{I_\ell} = \text{Ker } N_\sigma(-1)$ and $H^1(I_\ell, L) = L_{I_\ell}(-1) = \text{Coker } N_\sigma(-1)$ have the same class in $K^\sim(Y, G, F)$.

Remark. Note that the filtration $M_i L$ in the proof does not depend on the choice of σ . In fact, if $\tau = \sigma^r$ is another topological generator of $\mathbb{Z}_\ell(1)$, $r \in \mathbb{Z}_\ell^\times$, then

$$\bar{r} N_\tau(\bar{\sigma} \otimes a) = N_\tau(\bar{\tau} \otimes a) = (\tau - 1)a = [(1 + u)^r - 1]a = \bar{r}ua + u^2 Q(u)a,$$

where $Q(u)$ is a polynomial in $u = \sigma - 1$, hence

$$(N_\tau - N_\sigma)(\bar{\sigma} \otimes a) = (\bar{r})^{-1} u^2 Q(u)a \in \text{Im } N_\sigma^2.$$

To state an analogue of 2.2 for F -sheaves, we need a condition on inertia.

Proposition 2.8. *Let S be a quasi-separated scheme, $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Deligne-Mumford S -stacks, $m \in \mathbb{Z}$ be an integer. The following two conditions are equivalent:*

- (a) *For any algebraically closed field Ω and any point $x \in \mathcal{X}(\Omega)$, the order of the group $\text{Ker}(\text{Aut}_{\mathcal{X}}(x) \rightarrow \text{Aut}_{\mathcal{Y}}(y))$ is prime to m , where $y \in \mathcal{Y}(\Omega)$ is the image of x under f ;*
- (b) *For any algebraically closed field Ω and any point $y \in \mathcal{Y}(\Omega)$ and any lifting $x \in \mathcal{X}(\Omega)$ of y , the order of the group $\text{Aut}_{\mathcal{X}_y}(x)$ is prime to m .*

Note that the morphisms satisfying (a) are closed under composition while the morphisms satisfying (b) are closed under base change.

Definition 2.9. Morphisms satisfying the conditions of 2.8 are called *of prime to m inertia*.

Example 2.10. Let $(X, G) \rightarrow (Y, H)$ be an equivariant morphism of S -schemes (the finite groups G and H acting trivially on S). If for all geometric points $x \rightarrow X$, of image $y \rightarrow Y$, the order of the group $\text{Ker}(G_x \rightarrow H_y)$ is prime to m , where G_x and H_y are the inertia groups, then the induced morphism $[X/G] \rightarrow [Y/H]$ of quotient S -stacks is of prime to m inertia. Indeed, $\text{Aut}_{[X/G]}(\xi) = G_x$, $\text{Aut}_{[Y/H]}(v) = H_y$, where ξ is the composition $x \rightarrow X \rightarrow [X/G]$, v is the composition $y \rightarrow Y \rightarrow [Y/H]$.

We will deduce 2.8 in 2.16 from some general facts about inertia. Let \mathcal{C} be a 2-category. The 2-commutative squares in \mathcal{C} form a 2-category \mathcal{C}^\square in an obvious way. Let S be the partially ordered set

$$\{0, 1\}^2 = \{(i, j) \mid 0 \leq i, j \leq 1\}.$$

A pseudofunctor $F: S \rightarrow \mathcal{C}$ is a 2-commutative diagram in \mathcal{C} of the form

$$(2.10.1) \quad \begin{array}{ccc} F_{00} & \longrightarrow & F_{01} \\ \downarrow & \searrow & \downarrow \\ F_{10} & \longrightarrow & F_{11} \end{array}$$

The 2-category \mathcal{C}^\square can be identified with the 2-subcategory of the 2-category of pseudofunctors $S \rightarrow \mathcal{C}$, spanned by those pseudofunctors F for which the lower-left triangle is strictly commutative. A *2-Cartesian square* is a 2-commutative square of the form (2.10.1) which is a 2-limit diagram, namely which exhibits F_{00} as the 2-limit of the diagram indexed by $T = ((1, 0) \rightarrow (1, 1) \leftarrow (0, 1)) \subset S$.

Lemma 2.11. *Consider a 2-commutative square*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in \mathcal{C}^\square . Suppose that the squares $A_{ij}B_{ij}X_{ij}Y_{ij}$, $0 \leq i, j \leq 1$ and the squares B , X , Y are 2-Cartesian. Then the square A is 2-Cartesian.

Proof. The restriction $X|T \rightarrow Y|T \leftarrow B|T$ of the given square corresponds to a diagram in \mathcal{C} indexed by $T \times T'$, which we denote by D . Here $T' = T$. Then A_{00} is $\lim_{T'} \lim_T D$, where \lim stands for 2-limit. The assertion then follows from the canonical identification of $\lim_T \lim_{T'} D$ and $\lim_{T'} \lim_T D$. \square

Corollary 2.12. *Let*

$$\begin{array}{ccccc} & & A' & \xrightarrow{\quad} & A \\ & \swarrow & \downarrow & & \swarrow \\ B' & \xrightarrow{\quad} & B & & \\ \downarrow & & \downarrow & & \downarrow \\ & \swarrow & X' & \xrightarrow{\quad} & X \\ Y' & \xrightarrow{\quad} & Y & & \end{array}$$

be a 2-commutative cube in a 2-category. If the front, back and bottom squares are 2-Cartesian, then the top square is also 2-Cartesian.

Proof. It suffices to apply 2.11 to the square of squares

$$\begin{array}{ccc} A'AB'B & \longrightarrow & AAB B \\ \downarrow & & \downarrow \\ X'XY'Y & \longrightarrow & XXY Y \end{array}$$

\square

2.13. Let \mathcal{C} be a 2-category. We say that a morphism $f: X \rightarrow Y$ in \mathcal{C} is *faithful* if for every object W of \mathcal{C} , the functor

$$\mathrm{Hom}_{\mathcal{C}}(W, X) \rightarrow \mathrm{Hom}_{\mathcal{C}}(W, Y)$$

is faithful. Assume \mathcal{C} admits 2-fiber products. For any morphism $f: X \rightarrow Y$ in \mathcal{C} , we define the *inertia* of f to be

$$(2.13.1) \quad I_f = X \times_{\Delta_f, X \times_Y X, \Delta_f} X.$$

The two projections $X \times_Y X \rightarrow X$ induce two isomorphisms between the two projections $I_f \rightarrow X$. To fix ideas, we endow I_f with the first projection to X , which is a faithful morphism because it admits the diagonal morphism $\delta_f: X \rightarrow I_f$ as a section. For a morphism $g: W \rightarrow X$ in \mathcal{C} , $\text{Hom}_X(W, I_f)$ is equivalent to the group $\text{Aut}_{\mathcal{D}}(g)$, with the diagonal section $\delta_f g$ corresponding to the identity element of the group. Here \mathcal{D} is the category $\text{Hom}_Y(W, X)$. In the case where \mathcal{C} is the 2-category of categories fibered over a given category \mathcal{A} , an explicit description of I_f can be found in [The, 034H].

Lemma 2.14. *Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

be a 2-Cartesian square in a 2-category admitting 2-fiber products. Then the square

$$\begin{array}{ccc} I_{X'/Y'} & \longrightarrow & I_{X/Y} \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

is 2-Cartesian.

Proof. It suffices to apply 2.12 successively to the following cubes:

$$\begin{array}{ccccc} & & X' \times_{Y'} X' & \longrightarrow & X' \\ & \swarrow & \downarrow & & \swarrow \\ X \times_Y X & \longrightarrow & X & & X \\ \downarrow & & \downarrow & & \downarrow \\ & \swarrow & X' & \longrightarrow & Y' \\ & \swarrow & \downarrow & & \swarrow \\ X & \longrightarrow & Y & & Y' \end{array}$$

$$\begin{array}{ccccc} & & X' & \longrightarrow & X \\ & \swarrow & \parallel & & \swarrow \\ X' \times_{Y'} X' & \longrightarrow & X \times_Y X & & X \\ \downarrow & & \downarrow & & \downarrow \\ & \swarrow & X' & \longrightarrow & X \\ & \swarrow & \downarrow & & \swarrow \\ X' & \longrightarrow & X & & X \end{array}$$

$$\begin{array}{ccccc}
& & I_{X'/Y'} & \xrightarrow{\quad} & X' \\
& \swarrow & \downarrow & & \swarrow \\
I_{X/Y} & \xrightarrow{\quad} & X & & \\
\downarrow & & \downarrow & & \downarrow \\
& & X' & \xrightarrow{\quad} & X' \times_{Y'} X' \\
& \swarrow & \downarrow & & \swarrow \\
X & \xrightarrow{\quad} & X \times_Y X & &
\end{array}$$

□

Lemma 2.15. *Let $X \rightarrow Y \rightarrow Z$ be a sequence of morphisms in a 2-category admitting 2-fiber products. Then the canonical morphism $I_{X/Y} \rightarrow I_{X/Z}$ induces an isomorphism*

$$I_{X/Y} \xrightarrow{\sim} K = \text{Ker}(I_{X/Z} \rightarrow X \times_Y I_{Y/Z}).$$

Here K is defined by the following 2-commutative diagram with 2-Cartesian squares

$$\begin{array}{ccccc}
K & \xrightarrow{\quad} & X & \xrightarrow{\quad} & Y \\
\downarrow & & \downarrow & & \downarrow \delta_{Y/Z} \\
I_{X/Z} & \xrightarrow{\quad} & X \times_Y I_{Y/Z} & \xrightarrow{\quad} & I_{Y/Z}
\end{array}$$

Proof. Applying 2.11 successively to the squares of squares

$$\begin{array}{ccc}
X \times_Y X, Y, X \times_Z X, Y \times_Z Y & \xrightarrow{\quad} & XYXY \\
\downarrow & & \downarrow \\
XYXY & \xrightarrow{\quad} & YYZZ
\end{array}$$

and

$$\begin{array}{ccc}
I_{X/Y}, Y, I_{X/Z}, I_{Y/Z} & \xrightarrow{\quad} & XYXY \\
\downarrow & & \downarrow \\
XYXY & \xrightarrow{\quad} & X \times_Y X, Y, X \times_Z X, Y \times_Z Y
\end{array}$$

we see that the outer square of the 2-commutative diagram

$$\begin{array}{ccccc}
I_{X/Y} & \xrightarrow{\quad} & X & \xrightarrow{\quad} & Y \\
\downarrow & & \downarrow & & \downarrow \\
I_{X/Z} & \xrightarrow{\quad} & X \times_Y I_{Y/Z} & \xrightarrow{\quad} & I_{Y/Z}
\end{array}$$

is 2-Cartesian. Since the square on the right of the above diagram is 2-Cartesian, it follows from 2.12 that the square on the left is also 2-Cartesian. □

2.16. *Proof of 2.8.* By 2.15,

$$I_{\mathcal{X}/\mathcal{Y}} \simeq \text{Ker}(I_{\mathcal{X}/S} \rightarrow \mathcal{X} \times_{\mathcal{Y}} I_{\mathcal{Y}/S}).$$

By 2.14,

$$I_{\mathcal{X}_y/\mathcal{Y}} \simeq y \times_{\mathcal{Y}} I_{\mathcal{X}/\mathcal{Y}}$$

for $y \in \mathcal{Y}(\Omega)$. Therefore,

$$\text{Aut}_{\mathcal{X}_y}(x) \simeq \text{Hom}_{\mathcal{X}_y}(x, I_{\mathcal{X}_y/\mathcal{Y}}) \simeq \text{Hom}_{\mathcal{X}}(x, I_{\mathcal{X}/\mathcal{Y}}) \simeq \text{Ker}(\text{Aut}_{\mathcal{X}}(x) \rightarrow \text{Aut}_{\mathcal{Y}}(y))$$

for $x \in \mathcal{X}(\Omega)$ lifting y . □

2.17. Now let S be as in 2.1, $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of finite type Deligne-Mumford S -stacks of prime to ℓ inertia. Recall that F is a field of characteristic ℓ . Then we have functors [Zhe11, § 2]

$$Rf_*, Rf_! : D_c^b(\mathcal{X}, F) \rightarrow D_c^b(\mathcal{Y}, F).$$

They induce homomorphisms

$$f_*, f_! : K(\mathcal{X}, F) \rightarrow K(\mathcal{Y}, F)$$

and

$$f_*, f_! : K^\sim(\mathcal{X}, F) \rightarrow K^\sim(\mathcal{Y}, F)$$

by passing to quotients.

Theorem 2.18. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of finite type Deligne-Mumford S -stacks of prime to ℓ inertia. For any $x \in K(\mathcal{X}, F)$, $f_*(x^\sim) = f_!(x^\sim)$ in $K^\sim(\mathcal{Y}, F)$.*

The proof follows the same line as the proof of 2.2. In 2.6, up to shrinking X , we may assume in addition that G/G_0 acts freely on X , where $G_0 = \text{Ker}(G \rightarrow \text{Aut}(X))$. As before, by [Zhe09, 5.1], we are reduced to showing 2.18 in the case where $f: [X/G] \rightarrow [Y/H]$ is induced by an equivariant morphism $(X, G) \rightarrow (Y, H)$ of affine schemes of finite type over S , with G/G_0 acting freely on X . Then f is the composite morphism

$$[X/G] \rightarrow [(X/G_0 \cap N)/(G/G_0 \cap N)] \xrightarrow{g} [(X/N)/(G/N)] \rightarrow [Y/H],$$

where $N = \text{Ker}(G \rightarrow H)$. Since $N/G_0 \cap N$ acts freely on X , g is an isomorphism. By assumption, $G_0 \cap N$ has order prime to ℓ , hence we are reduced to showing 2.18 in the case where $f: [X/G] \rightarrow [Y/H]$ is induced by an equivariant morphism $(X, G) \rightarrow (Y, H)$ of affine schemes of finite type over S , with ℓ prime to the order of $\text{Ker}(G \rightarrow H)$. We prove the analogues of 1.8, 1.3 and 2.4 as before, with 1.9 replaced by 2.7. (For the analogues of 1.3 and 2.4, we assume that $\text{Ker}(G \rightarrow H)$ has order prime to ℓ .)

3 Free actions and vanishing theorems: ℓ -adic and Betti cohomologies

3.1. Let k be an algebraically closed field of characteristic exponent p , ℓ a prime number $\neq p$, X an algebraic k -space separated and of finite type, endowed with an action of a finite group G . We extend the notations $t_\ell(s)$, $t_{c,\ell}(s)$ and $\chi(X, G, \mathbb{Q}_\ell)$ ((0.1) through (0.3)) to this situation. By 2.2, $t_\ell(s) = t_{c,\ell}(s)$ for $s \in G$.

Theorem 3.2. *Under the assumptions of 3.1, we have:*

- (a) *For $s \in G$, $t_\ell(s)$ is an integer $t(s)$ independent of ℓ ;*
- (b) *If G acts freely on X , $R\Gamma_c(X, \mathbb{Z}_\ell)$ (resp. $R\Gamma(X, \mathbb{Z}_\ell)$) is a perfect complex of $\mathbb{Z}_\ell[G]$ -modules (i.e. is isomorphic to a bounded complex of finitely generated projective $\mathbb{Z}_\ell[G]$ -modules);*
- (c) *If G acts freely on X , then $t(s) = 0$ for every $s \in G$ whose order is not a power of p .*

Proof. Assertion (a) for $t_{c,\ell}(s)$ in the case of a separated scheme is a result of Deligne-Lusztig [DL76, 3.3]. The general case is similar: by additivity of $t_{c,\ell}$ with respect to X , we may assume X affine; by spreading out one reduces to the case where k is the algebraic closure of a finite field \mathbb{F}_q and X with its action of G is defined over \mathbb{F}_q ; in this case, if F is the geometric Frobenius of $\text{Spec } k / \text{Spec } \mathbb{F}_q$, for any $n \geq 1$, $s \times_{\mathbb{F}_q} F^n$ is identified with $\text{Id}_{X'_n} \times_{\mathbb{F}_{q^n}} F^n$ for a suitable X'_n / \mathbb{F}_{q^n} , and the assertion follows from Grothendieck's trace formula.

The argument for (b) and (c) is analogous to that of [Ill81, proof of 2.5]. We have

$$(3.2.1) \quad R\Gamma(X, \mathbb{Z}_\ell) = R\Gamma(X/G, \pi_* \mathbb{Z}_\ell),$$

where $\pi: X \rightarrow X/G$ is the projection. Here X/G is an algebraic space of finite type over k . As π is an étale Galois cover of group G , $\pi_* \mathbb{Z}_\ell$ is a lisse sheaf, locally free of rank one over $\mathbb{Z}_\ell[G]$. For any $\mathbb{Z}_\ell[G]$ -module M of finite type, we have, by the projection formula,

$$(3.2.2) \quad R\Gamma(X/G, \pi_* \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell[G]}^L M \xrightarrow{\sim} R\Gamma(X/G, \pi_* \mathbb{Z}_\ell \otimes_{\mathbb{Z}_\ell[G]} M),$$

which implies that $R\Gamma(X/G, \pi_* \mathbb{Z}_\ell)$ is of finite tor-dimension, hence perfect (as $R\Gamma(X/G, \pi_* \mathbb{Z}_\ell)$ belongs to $D_c^b(\mathbb{Z}_\ell)$ by (3.2.1)). (This type of argument seems to have appeared for the first time in Grothendieck's proof of the Euler-Poincaré and Lefschetz formulas for curves, cf. [SGA 5, III, X], [SGA 4½, Rapport].) The proof for $R\Gamma_c$ is analogous, except that we may assume X/G to be a separated scheme by induction on $\dim X$.

By (a) the character t of $\chi_c(X, G, \mathbb{Q}_\ell)$ ($= \chi(X, G, \mathbb{Q}_\ell)$ (1.4.2)) has values in \mathbb{Z} and is independent of ℓ . Therefore (c) follows from the theory of modular characters [Ser98, th. 36, p. 145]: if P is a finitely generated projective $\mathbb{Z}_\ell[G]$ -module, the character of $P \otimes \mathbb{Q}_\ell$ vanishes on ℓ -singular elements of G , i.e. elements whose order is divisible by ℓ . \square

For $s = 1$, the fact that $t_c(1) = \chi_c(X, \mathbb{Q}_\ell)$ is independent of ℓ had been known since the early 1960s, as it is an immediate consequence of Grothendieck's trace formula (cf. [Ill06, § 1]). Thanks to Gabber's theorem [Fuj02], one can show the independence of ℓ for $t_\ell(s)$ independently of 1.4.

As observed in [DL76, 3.12], 3.2 implies:

Corollary 3.3. *If G acts freely on X and the order of G is prime to p , then, with the notations of 1.4,*

$$(3.3.1) \quad \chi(X, G, \mathbb{Q}_\ell) = \chi(X/G) \text{Reg}_{\mathbb{Q}_\ell}(G),$$

where $\text{Reg}_{\mathbb{Q}_\ell}(G)$ is the class of the regular representation and

$$\chi(X/G) = \sum (-1)^i \dim H_c^i(X/G, \mathbb{Q}_\ell) = \sum (-1)^i \dim H^i(X/G, \mathbb{Q}_\ell)$$

is the Euler-Poincaré characteristic of the algebraic space X/G .

Proof. Indeed, by 3.2, we have $t(s) = 0$ for all $s \neq 1$. Therefore, by [Ser98, 2.4, 12.1], there exists $m \in \mathbb{Z}$ such that

$$\chi(X, G, \mathbb{Q}_\ell) = m \text{Reg}_{\mathbb{Q}_\ell}(G).$$

As, by (3.2.2),

$$R\Gamma(X/G, \pi_* \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell[G]}^L \mathbb{Z}_\ell = R\Gamma(X/G, \mathbb{Z}_\ell),$$

one finds $m = \chi(X/G, \mathbb{Q}_\ell)$. \square

The following application was suggested to the first author by Serre:

Corollary 3.4. *With the notations of 3.1, assume that G is cyclic, generated by s . Assume moreover that the order of G is prime to p and that s has no fixed points. Then $t(s) = 0$.*

Proof. When G acts freely, this is a particular case of 3.3. In the general case, for any subgroup H of G , denote by X^H the fixed point set of G (a closed algebraic subset of X) and, as in [Ver73, § 2], by X_H the open subset of X^H defined by

$$X_H = X^H - \bigcup_{H' \supset H, H' \neq H} X^{H'}.$$

Each X^H (resp. X_H) is G -stable, and the inertia group at any point of X_H is H . On X_H the quotient G/H acts freely. As X is the disjoint union of the X_H 's for H running through the subgroups of G , we have

$$t(s) = \sum_H \text{Tr}(s, H_c^*(X_H, \mathbb{Q}_\ell)).$$

As s generates G/H , $\text{Tr}(s, H_c^*(X_H, \mathbb{Q}_\ell)) = 0$, hence $t(s) = 0$. \square

Finally, here is an application to Betti cohomology:

Corollary 3.5. *Let X/\mathbb{C} be an algebraic space separated and of finite type over \mathbb{C} endowed with a free action of a finite group G . Let $R_{\mathbb{Q}}(G)$ denote the Grothendieck group of \mathbb{Q} -linear finite dimensional representations of G , and*

$$\begin{aligned} \chi_c(X, G, \mathbb{Q}) &= \sum (-1)^i [H_c^i(X, \mathbb{Q})] \in R_{\mathbb{Q}}(G), \\ \chi(X, G, \mathbb{Q}) &= \sum (-1)^i [H^i(X, \mathbb{Q})] \in R_{\mathbb{Q}}(G), \end{aligned}$$

where $[-]$ denotes a class in $R_{\mathbb{Q}}(G)$. Then:

$$(3.5.1) \quad \chi_c(X, G, \mathbb{Q}) = \chi(X, G, \mathbb{Q}) = \chi(X/G, \mathbb{Q}) \text{Reg}_{\mathbb{Q}}(G),$$

where $\text{Reg}_{\mathbb{Q}}(G)$ is the class of the regular representation and

$$\chi_c(X/G, \mathbb{Q}) = \chi(X/G, \mathbb{Q})$$

is the Euler-Poincaré characteristic of X/G . (Here by $H_c^i(Y, \mathbb{Q})$ (resp. $H^i(Y, \mathbb{Q})$), for Y/\mathbb{C} an algebraic space separated and of finite type, we mean $H_c^i(Y(\mathbb{C}), \mathbb{Q})$ (resp. $H^i(Y(\mathbb{C}), \mathbb{Q})$), where $Y(\mathbb{C})$ is the space of rational points of Y with the classical topology defined in [Art69, 1.6].)

Recall the comparison theorem between étale and Betti cohomologies for algebraic spaces: for a morphism $f: Y \rightarrow Z$ of finite type algebraic spaces over \mathbb{C} , we have a 2-commutative square of topoi

$$\begin{array}{ccc} Y(\mathbb{C})_{\text{cl}} & \longrightarrow & Y_{\text{et}} \\ \downarrow & & \downarrow \\ Z(\mathbb{C})_{\text{cl}} & \longrightarrow & Z_{\text{et}} \end{array}$$

and the base change morphism $Rf_{\text{cl}*}\epsilon_Z^*\mathcal{F} \rightarrow \epsilon_Y^*Rf_{\text{et}*}\mathcal{F}$ is an isomorphism for any constructible torsion abelian sheaf \mathcal{F} . See [SGA 4, XVI 4.1] for the case of schemes. The general case follows from this case: the problem being local, we may assume Z to be a scheme; then we take an étale cover of Y by schemes and apply cohomological descent. In particular, $\chi(Y, \mathbb{Q}) = \chi(Y, \mathbb{Q}_\ell)$ (see 2.17 for finiteness). We also have $\chi_c(Y, \mathbb{Q}) = \chi_c(Y, \mathbb{Q}_\ell)$ for Y separated. Indeed, it suffices to take a stratification by separated schemes and apply the identity to each stratum [SGA 4 $\frac{1}{2}$, Arcata IV.6.3].

The homomorphism $R_{\mathbb{Q}}(G) \rightarrow R_{\mathbb{Q}_\ell}(G)$ is injective [Ser98, 14.6], and by the comparison theorem, $\chi_c(X, G, \mathbb{Q}_\ell)$ (resp. $\chi(X, G, \mathbb{Q}_\ell)$) is the image of $\chi_c(X, G, \mathbb{Q})$ (resp. $\chi(X, G, \mathbb{Q})$). Therefore (3.5.1) follows from (3.3.1).

Remark 3.6. (a) The equality

$$(3.6.1) \quad \chi_c(X, G, \mathbb{Q}) = \chi_c(X/G)\text{Reg}_{\mathbb{Q}}(G)$$

was established by Verdier [Ver73, lemme, p. 443], using a similar method. At the time, the equality $\chi = \chi_c$ was unknown. Actually, Verdier proves (3.5.1) more generally for locally compact spaces X which are of *finite topological dimension* (i.e. such that there exists an integer N such that $H_c^n(X, F) = 0$ for all $n > N$ and all abelian sheaves F) and *cohomologically of finite type* (i.e. such that, for each n , $H_c^n(X, \mathbb{Z})$ is finitely generated), endowed with a continuous and free action of a finite group G (X/G is then also cohomologically of finite type). When the action of G is no longer assumed to be free, but the fixed point set X^H is cohomologically of finite type for every subgroup H of G , using a decomposition of X of the type considered in the proof of 3.4, Verdier formally deduces from (3.5.1) a general formula for $\chi_c(X, G, \mathbb{Q})$ as a linear combination of certain induced representations (*loc. cit.*, p. 443). We will come back to this and ℓ -adic variants in § 4.

(b) By a theorem of Deligne reported on in [Ill81], if X is a separated scheme, (3.3.1) holds more generally if the action of G is only assumed to be *tame at infinity*. We will discuss this and related results in § 4.

3.7. Let k be a field, \bar{k} a separable closure of k , X an algebraic k -space of finite type, endowed with a free action of a group G of order ℓ^n , where ℓ is a prime different from the characteristic of k . Serre proved (in the case of a separated scheme) that, under these assumptions, for any $g \in \Gamma = \text{Gal}(\bar{k}/k)$, the ℓ -adic integer $\text{Tr}(g, H_c^*(X_{\bar{k}}, \mathbb{Q}_\ell))$ is divisible by ℓ^n ([Ser07], [Ill06, 7.5]). On the other hand, by 3.2 (c), for $s \in G$, $s \neq 1$, $\text{Tr}(s, H_c^*(X_{\bar{k}}, \mathbb{Q}_\ell)) = 0$. More generally:

Theorem 3.8. *Under the assumptions of 3.7, for any $s \in G$ and any $g \in \Gamma$, we have*

$$(3.8.1) \quad \text{Tr}(sg, H_c^*(X_{\bar{k}}, \mathbb{Q}_\ell)) \equiv 0 \pmod{\ell^m},$$

where ℓ^m is the order of the centralizer of s in G .

The proof follows the pattern of that of Serre's theorem (*loc. cit.*). We may assume X affine. Choose a model \mathcal{X}/S of X/k where S is the spectrum of a finitely generated, integral, normal, sub \mathbb{Z} -algebra of k , \mathcal{X}/S affine and of finite type, endowed with a free action of G by S -automorphisms, such that X/k , with its G -action, comes from \mathcal{X}/S by base change. Suppose that there exists a pair (s, g) such that $t_c(sg) \not\equiv 0 \pmod{\ell^m}$, where $t_c(sg) = \text{Tr}(sg, H_c^*(X_{\bar{k}}, \mathbb{Q}_\ell))$. Applying Chebotarev's generalized density theorem as in [Ill06, proof of 7.1, (1) \Rightarrow (2)], one finds a point y of S with value in a finite field \mathbb{F}_q of characteristic $p \neq \ell$ such that $t_c(sg) \equiv \text{Tr}(sF, H_c^*(\mathcal{X}_{\bar{y}}, \mathbb{Q}_\ell)) \pmod{\ell^m}$, where \bar{y} is an algebraic geometric point over y and F the geometric Frobenius automorphism of \bar{y}/y . By Deligne-Lusztig (*loc. cit.*), sF is the geometric Frobenius automorphism of $X'_{\bar{y}}$ for some X'/\mathbb{F}_q , therefore, by Grothendieck's trace formula, the trace of sF is the cardinality of the set E of rational points x of $\mathcal{X}_{\bar{y}}$ such that $sFx = x$. Since the action of G commutes with F , the centralizer of s in G acts freely on E , hence this cardinality is divisible by ℓ^m , which is a contradiction.

Remark 3.9. (a) For $s \neq 1$, one can't expect $\text{Tr}(sg, H_c^*(X_{\bar{k}}, \mathbb{Q}_\ell)) = 0$ for all $g \in \Gamma$. Serre gives the following example: let $k = \mathbb{F}_q$ with $q - 1 \equiv 0 \pmod{\ell^n}$, and let $X = (\mathbb{G}_m)_k$; let s be the translation in X by a rational point of order ℓ^n (a primitive ℓ^n -th root of 1 in k). Then the trace of sF is equal to the trace of F , i.e. $q - 1$.

(b) We don't know whether 3.8 holds with H_c^* replaced by H^* .

3.10. Under the assumptions of 3.1, when G is not assumed to act freely, $R\Gamma(X, \mathbb{Z}_\ell)$ (resp. $R\Gamma_c(X, \mathbb{Z}_\ell)$), as an object of $D_c^b(\mathbb{Z}_\ell[G])$ is not, in general, a perfect complex. It belongs, however, to a certain full subcategory of $D_c^b(\mathbb{Z}_\ell[G])$ considered by Rickard [Ric94].

Recall that a $\mathbb{Z}_\ell[G]$ -module is called a *permutation module* if it is isomorphic to a module of the form $\mathbb{Z}_\ell[E]$ for a finite G -set E , in other words, if it admits a finite basis over \mathbb{Z}_ℓ which is set-theoretically stable under G . Denote by $\mathbb{Z}_\ell[G]_{\text{perm}}$ the full subcategory of the category of $\mathbb{Z}_\ell[G]$ -modules consisting of direct summands of permutation modules. This is an additive subcategory. By a result of Rouquier [Rou], the natural functor $K^b(\mathbb{Z}_\ell[G]_{\text{perm}}) \rightarrow D_c^b(\mathbb{Z}_\ell[G])$ induces a fully faithful functor

$$(3.10.1) \quad K^b(\mathbb{Z}_\ell[G]_{\text{perm}})/K_0^b(\mathbb{Z}_\ell[G]_{\text{perm}}) \rightarrow D_c^b(\mathbb{Z}_\ell[G]),$$

where $K_0^b(\mathbb{Z}_\ell[G]_{\text{perm}})$ denotes the full subcategory of $K^b(\mathbb{Z}_\ell[G]_{\text{perm}})$ consisting of acyclic complexes. In particular, the essential image $D^b(\mathbb{Z}_\ell[G])_{\text{perm}}$ of (3.10.1) is a triangulated subcategory of $D_c^b(\mathbb{Z}_\ell[G])$.

Theorem 3.11. *Under the assumptions of 3.1, denote by $D^b(\mathbb{Z}_\ell[G])_{X\text{-perm}}$ the smallest full triangulated subcategory of $D^b(\mathbb{Z}_\ell[G])_{\text{perm}}$ containing the direct summands of permutation modules of the form $\mathbb{Z}_\ell[G/I]$, where I runs through the inertia groups of G . Then $R\Gamma_c(X, \mathbb{Z}_\ell)$ (resp. $R\Gamma(X, \mathbb{Z}_\ell)$, if X is separated,) belongs to $D^b(\mathbb{Z}_\ell[G])_{X\text{-perm}}$.*

Remark 3.12. (a) For the result on $R\Gamma(X, \mathbb{Z}_\ell)$, the assumption that X is separated serves only to guarantee the existence of X/G . One may replace it by the weaker assumption that the inertia subgroup $I(G, X)$ of $G \times X$ is finite over X .

(b) When G acts freely, $D^b(\mathbb{Z}_\ell[G])_{X\text{-perm}}$ is the category of perfect complexes of $\mathbb{Z}_\ell[G]$ -modules: one recovers 3.2 (b).

(c) When X is a separated scheme and G acts admissibly, the result of 3.11 for $R\Gamma_c(X, \mathbb{Z}_\ell)$ is a weak form of Rickard's theorem [Ric94, 3.2]. Rickard actually constructs a bounded

complex of finite sums of direct summands of permutation modules of the form $\mathbb{Z}_\ell[G/I]$, where I is an inertia group of G , well defined up to homotopy, representing $R\Gamma_c(X, \mathbb{Z}_\ell)$. Rickard does not consider the case of $R\Gamma$.

3.13. Let us introduce some notations before proving 3.11. For a homomorphism of finite groups $f: G \rightarrow H$, the contracted product is defined to be $X \wedge^G H = (X \times H)/G$, where G acts on $X \times H$ by $(x, h)g = (xg, f(g)^{-1}h)$. See [Gir71, III 1.3.1] for a definition in a topos. The right translation of H induces an action of H on $X \wedge^G H$. We have $(X \wedge^G H) \wedge^H \{1\} \simeq X \wedge^G \{1\}$, namely $(X \wedge^G H)/H \simeq X/G$. If f is a monomorphism, the projection $X \wedge^G H \rightarrow X$ is a finite étale cover of fibers isomorphic to H/G .

As in the proof of 3.4, we denote by X_H the open subset of X^H which is the complement of the union of the $X^{H'}$ for H' strictly containing H . The stabilizer of any geometric point in X_H is H . Let \mathcal{S} be the set of conjugacy classes of subgroups of G . For $S \in \mathcal{S}$, denote by X_S the (disjoint) union of the X_H 's for $H \in S$, with its induced action of G , and $Y_S = X_S/G$. For $H \in S$, the normalizer $N_G(H)$ acts on X_H through $N_G(H)/H$. The projection $X_H \rightarrow X_H/N_G(H) = Y_S$ is an étale Galois cover of group $N_G(H)/H$, and $X_S = X_H \wedge^{N_G(H)} G$. The Y_S 's, for $S \in \mathcal{S}$, form a decomposition of $Y = X/G$ into disjoint locally closed algebraic subspaces of Y .

Proof of 3.11. Let us first treat the case of $R\Gamma_c$. We may assume X separated. We proceed by Noetherian induction on X . There is a G -stable dense open subset V of X which is a disjoint union of irreducible schemes. Take one component W of V , and let H be the inertia group at the generic point. Then $W^H = W$ as sets. Let $U = W_S$, where S is the conjugacy class of H . Then U is a nonempty G -stable open subset of V , disjoint union of the $U_{H'} = U^{H'}$, for H' running through the conjugacy class S , such that G acts transitively on the maximal points of U . Up to shrinking U , we may assume U affine. We have a distinguished triangle in $D_c^b(\mathbb{Z}_\ell[G])$:

$$R\Gamma_c(U, \mathbb{Z}_\ell) \rightarrow R\Gamma_c(X, \mathbb{Z}_\ell) \rightarrow R\Gamma_c(Y, \mathbb{Z}_\ell) \rightarrow,$$

where Y is the complement of U in X . By the induction hypothesis, we may therefore assume that $X = U$. The group $N_G(H)$ acts on X_H . The natural map (of $D^b(\mathbb{Z}_\ell[G])$)

$$\mathbb{Z}_\ell[G] \otimes_{\mathbb{Z}_\ell[N_G(H)]}^L R\Gamma_c(X_H, \mathbb{Z}_\ell) \rightarrow R\Gamma_c(X, \mathbb{Z}_\ell)$$

is an isomorphism, as $X = G \wedge^{N_G(H)} X_H$. As $N_G(H)$ acts on X_H through $N_G(H)/H$ and the action of $N_G(H)/H$ is free, $R\Gamma_c(X_H, \mathbb{Z}_\ell)$ is perfect over $\mathbb{Z}_\ell[N_G(H)/H]$ (3.1 (b)). As $\mathbb{Z}_\ell[G] \otimes_{\mathbb{Z}_\ell[N_G(H)]}^L \mathbb{Z}_\ell[N_G(H)/H] = \mathbb{Z}_\ell[G/H]$, it follows that $R\Gamma_c(X, \mathbb{Z}_\ell)$ is represented by a bounded complex of finite sums of direct summands of $\mathbb{Z}_\ell[G/H]$, which finishes the proof in this case.

The proof for the case of $R\Gamma$ is similar. Consider the open immersion $u: U \rightarrow X$ as above, and its complement $i: Y \rightarrow X$. The exact sequence $0 \rightarrow u_! \mathbb{Z}_\ell \rightarrow \mathbb{Z}_\ell \rightarrow i_* \mathbb{Z}_\ell \rightarrow 0$ gives an exact triangle in $D_c^b(\mathbb{Z}_\ell[G])$:

$$R\Gamma(X, u_! \mathbb{Z}_\ell) \rightarrow R\Gamma(X, \mathbb{Z}_\ell) \rightarrow R\Gamma(Y, \mathbb{Z}_\ell) \rightarrow .$$

By the induction hypothesis it suffices to show that $R\Gamma(X, u_! \mathbb{Z}_\ell)$ belongs to $D^b(\mathbb{Z}_\ell[G])_{X\text{-perm}}$.

Consider the commutative diagram

$$(3.13.1) \quad \begin{array}{ccc} U & \xrightarrow{u} & X \\ f_U \downarrow & & \downarrow f \\ U/G & \xrightarrow{v} & X/G \end{array},$$

where f is the canonical projection. We have, in $D_c^b(\mathbb{Z}_\ell[G])$ (by 2.17),

$$(3.13.2) \quad R\Gamma(X, u_! \mathbb{Z}_\ell) = R\Gamma(X/G, f_* u_! \mathbb{Z}_\ell) = R\Gamma(X/G, v_!(f_U)_* \mathbb{Z}_\ell).$$

As above, let H be a member of S , and let $f_H: U_H \rightarrow U/G$ be the restriction of f_U to U_H . We have $(f_U)_* \mathbb{Z}_\ell = \mathbb{Z}_\ell[G] \otimes_{\mathbb{Z}_\ell[N_G(H)]} (f_H)_* \mathbb{Z}_\ell$, hence $v_!(f_U)_* \mathbb{Z}_\ell = \mathbb{Z}_\ell[G] \otimes_{\mathbb{Z}_\ell[N_G(H)]} v_!(f_H)_* \mathbb{Z}_\ell$, so that by the projection formula, we get

$$(3.13.3) \quad R\Gamma(X/G, v_!(f_U)_* \mathbb{Z}_\ell) = \mathbb{Z}_\ell[G] \otimes_{\mathbb{Z}_\ell[N_G(H)]}^L R\Gamma(X/G, v_!(f_H)_* \mathbb{Z}_\ell).$$

As $N_G(H)$ acts on U_H through $N_G(H)/H$ and the action of $N_G(H)/H$ is free, $(f_H)_* \mathbb{Z}_\ell$ is locally isomorphic to $\mathbb{Z}_\ell[N_G(H)/H]$. Therefore $v_!(f_H)_* \mathbb{Z}_\ell$ is of finite tor-dimension over $N_G(H)/H$, and consequently, $R\Gamma(X/G, v_!(f_H)_* \mathbb{Z}_\ell)$ is perfect over $N_G(H)/H$. Then the conclusion follows from (3.13.2) and (3.13.3) by the same argument as for the case of $R\Gamma_c$. \square

4 Tameness at infinity

In this section we fix an algebraically closed field k of characteristic exponent p and a prime number $\ell \neq p$.

4.1. Let Y be a normal, connected scheme, separated and of finite type over k , G a finite group, and $f: X \rightarrow Y$ an étale Galois cover of group G . We have $Y = X/G$. Let \bar{Y} be a normal compactification of Y over k (i.e. a proper, normal, connected scheme over k containing Y as a dense open subset). We say that X is *tamely ramified along $\bar{Y} - Y$* if G acts *tamely* on the normalization \bar{X} of \bar{Y} in X , i.e. the inertia groups $G_{\bar{x}} \subset G$, stabilizers of geometric points \bar{x} of \bar{X} in G , are of order prime to p (cf. [Ill81, 2.6]). We say that X is *tamely ramified at infinity over Y/k* if there exists a normal compactification \bar{Y} of Y such that X is tamely ramified along $\bar{Y} - Y$. By [Ill81, 2.8], (3.3.1) still holds in this case, namely, we have an equality in the Grothendieck group $P_\ell(G) = K^\bullet(\mathbb{Z}_\ell[G])$ of finitely generated projective $\mathbb{Z}_\ell[G]$ -modules ($= K^\bullet(\mathbb{F}_\ell[G])$):

$$(4.1.1) \quad \chi_c(X, G, \mathbb{Z}_\ell) = \chi(X/G) \text{Reg}_{\mathbb{Z}_\ell}(G),$$

or, equivalently,

$$(4.1.2) \quad \chi_c(X, G, \mathbb{F}_\ell) = \chi(X/G) \text{Reg}_{\mathbb{F}_\ell}(G),$$

where $\chi_c(X, G, \mathbb{Z}_\ell) = \chi_c(X, G, \mathbb{F}_\ell)$ is the class of $R\Gamma_c(X, G, \mathbb{Z}_\ell)$ (or $R\Gamma_c(X, G, \mathbb{F}_\ell)$) in $P_\ell(G)$, and Reg denotes a regular representation. Note that, as the natural homomorphism $P_\ell(G) \rightarrow R_{\mathbb{Q}_\ell}(G)$ is injective, we have

$$(4.1.3) \quad \chi_c(X, G, \mathbb{Z}_\ell) = \chi(X, G, \mathbb{Z}_\ell) = \chi(X, G, \mathbb{F}_\ell) = \chi_c(X, G, \mathbb{F}_\ell),$$

where $\chi(X, G, \mathbb{Z}_\ell)$ (resp. $\chi(X, G, \mathbb{F}_\ell)$) is the class of $R\Gamma(X, G, \mathbb{Z}_\ell)$ (resp. $R\Gamma(X, G, \mathbb{F}_\ell)$).

4.2. We will reformulate the notion of tameness at infinity in terms of *Vidal's group* $K(Y, \mathbb{F}_\ell)_t^0$ [Vid04, 2.3.1] (where the group is denoted by $K_c(Y, \mathbb{F}_\ell)_t^0$).

Let us briefly recall its definition. Let Z be a scheme separated and of finite type over k . We denote by $K(Z, \mathbb{F}_\ell)$ the Grothendieck group of constructible \mathbb{F}_ℓ -modules on Z , and by $K_{\text{lis}}(Z, \mathbb{F}_\ell)$ the subgroup generated by the classes of lisse \mathbb{F}_ℓ -sheaves, and by $[\mathcal{F}]$ the class in $K(Z, \mathbb{F}_\ell)$ of a constructible sheaf \mathcal{F} . When Z is normal, connected, with geometric generic point ζ , $K_{\text{lis}}(Z, \mathbb{F}_\ell)$ is the Grothendieck group of finite, continuous $\mathbb{F}_\ell[\pi_1(Z, \bar{\zeta})]$ -modules. In this case, for $a \in K_{\text{lis}}(Z, \mathbb{F}_\ell)$, we denote by

$$\text{Tr}^{\text{Br}}(-, a): \pi_1(Z, \zeta) \rightarrow \mathbb{Z}_\ell, \quad g \mapsto \text{Tr}^{\text{Br}}(g, a_\zeta)$$

its Brauer trace (on ℓ -regular elements, extended by zero on ℓ -singular ones). Vidal defines a closed subset

$$E_{Z/k} \subset \pi_1(Z, \zeta),$$

stable under conjugation, called the *wild part* of $\pi_1(Z, \zeta)$: this is the intersection, over all normal compactifications \bar{Z} of Z , of the closures $E_{Z/k, \bar{Z}}$ of the unions of the conjugates of the images of the pro- p -Sylow subgroups of the local fundamental groups $\pi_1(\bar{Z}_{(y)} \times_{\bar{Z}} Z, z)$, where y runs through geometric points of $\bar{Z} - Z$ and z is a geometric point of $\bar{Z}_{(y)} \times_{\bar{Z}} Z$ above ζ (see [Vid04, 2.1], [Vid05, 1.1] for more details). Now, for Z only assumed to be separated and of finite type over k , Vidal defines

$$K(Z, \mathbb{F}_\ell)_t^0 \subset K(Z, \mathbb{F}_\ell)$$

as the subgroup generated by the classes $i_! a$, where $i: Y \rightarrow Z$ is separated and quasi-finite, with Y normal connected, and $a \in K_{\text{lis}}(Y, \mathbb{F}_\ell)$ has the property that, for all $g \in E_{Y/k}$, $\text{Tr}^{\text{Br}}(g, a) = 0$.

We extend this definition to algebraic spaces. More precisely, for an algebraic space Z of finite type over k , we denote by $K(Z, \mathbb{F}_\ell)$ the Grothendieck group of constructible \mathbb{F}_ℓ -modules on Z , and we define

$$K(Z, \mathbb{F}_\ell)_t^0 \subset K(Z, \mathbb{F}_\ell)$$

as the subgroup generated by the classes $i_! a$, where $i: Y \rightarrow Z$ is quasi-finite, with Y a separated normal connected scheme, and $a \in K_{\text{lis}}(Y, \mathbb{F}_\ell)$ has the property that, for all $g \in E_{Y/k}$, $\text{Tr}^{\text{Br}}(g, a) = 0$. This definition does not depend on the choice of geometric points.

Recall that, when Z is a separated normal connected scheme, it follows from a valuative criterion of Gabber that, for $a \in K_{\text{lis}}(Z, \mathbb{F}_\ell)$, a belongs to $K(Z, \mathbb{F}_\ell)_t^0$ if and only if there exists a normal compactification \bar{Z} of Z such that, for all $g \in E_{Z/k, \bar{Z}}$, $\text{Tr}^{\text{Br}}(g, a) = 0$ [Vid05, 6.2 (ii)].

The following is a variant of [Vid04, 2.3.3] and [Vid05, 0.1]:

Proposition 4.3. *Let $f: Z \rightarrow W$ be a morphism of algebraic spaces of finite type over k . Then*

- (a) *The map $f^*: K(W, \mathbb{F}_\ell) \rightarrow K(Z, \mathbb{F}_\ell)$ sends $K(W, \mathbb{F}_\ell)_t^0$ into $K(Z, \mathbb{F}_\ell)_t^0$.*
- (b) *The map $f_!: K(Z, \mathbb{F}_\ell) \rightarrow K(W, \mathbb{F}_\ell)$ sends $K(Z, \mathbb{F}_\ell)_t^0$ into $K(W, \mathbb{F}_\ell)_t^0$.*

(c) If $Z = \coprod_I Z_i$ is a partition of Z into locally closed algebraic subspaces, then $\phi: a \mapsto (a|Z_i)_I$ defines an isomorphism $K(Z, \mathbb{F}_\ell)_t^0 \simeq \oplus_I K(Z_i, \mathbb{F}_\ell)_t^0$.

(d) $K(Z, \mathbb{F}_\ell)_t^0$ is an ideal of the ring $K(Z, \mathbb{F}_\ell)$.

Proof. (a) Let X be a separated connected normal scheme of finite type over k , $i: X \rightarrow W$ be a quasi-finite morphism, $a \in K_{\text{lis}}(X, \mathbb{F}_\ell)$ satisfying $\text{Tr}^{\text{Br}}(g, a) = 0$ for all $g \in E_{X/k}$. Then, by base change,

$$(4.3.1) \quad f^* i_! a = \sum_{j \in J} i_{Y_j!} f_j^* a,$$

where $X \times_W Z = \coprod_{j \in J} Y_j$ is a partition into locally closed separated normal connected subschemes, $i_{Y_j}: Y_j \rightarrow Z$ and $f_j: Y_j \rightarrow X$ are the projections. Note that $f_j^* a \in K_{\text{lis}}(Y_j, \mathbb{F}_\ell)$. If we still denote by f_j the map $E_{Y_j, k} \rightarrow E_{X, k}$ induced by f_j [Vid04, 2.1.1], then

$$\text{Tr}^{\text{Br}}(g, f_j^* a) = \text{Tr}^{\text{Br}}(f_j(g), a) = 0.$$

Thus $f^* i_! a$ belongs to $K(W, \mathbb{F}_\ell)_t^0$ by (4.3.1).

(c) By (a), the homomorphism ϕ in (c) is well defined. We define a homomorphism $\psi: \oplus_I K(Z_i, \mathbb{F}_\ell)_t^0 \rightarrow K(Z, \mathbb{F}_\ell)_t^0$ by $(a|Z_i)_I \mapsto \sum_I i_{Z_i!} a|Z_i$, where i_{Z_i} is the immersion $Z_i \rightarrow Z$. The two homomorphisms are clearly inverse to each other.

(b) If f is quasi-finite, (b) follows from the definition. For the general case, applying the quasi-finite case and (c), we may reduce to the case where f is morphism of separated schemes. In this case (b) is [Vid05, 0.1].

(d) Let $i_X: X \rightarrow Z$ be a quasi-finite morphism where X is a separated normal connected scheme, let $a \in K_{\text{lis}}(X, \mathbb{F}_\ell)$ satisfying $\text{Tr}^{\text{Br}}(g, a) = 0$ for all $g \in E_{X/k}$, and let $b \in K(Z, \mathbb{F}_\ell)$. By projection formula,

$$(4.3.2) \quad (i_! a)b = i_!(a(i^* b)) = \sum_{j \in J} i_{X_j!}((a|X_j)(b|X_j)),$$

where $X = \coprod_{j \in J} X_j$ is a partition into locally closed normal connected subschemes such that $b|X_j \in K_{\text{lis}}(X_j, \mathbb{F}_\ell)$, $i_{X_j}: X_j \rightarrow Z$ is the composition with i . As in (a), it follows from the functoriality of E that $\text{Tr}^{\text{Br}}(g, a|X_j) = 0$ for all $g \in E_{X_j/k}$. Thus $\text{Tr}^{\text{Br}}(g, (a|X_j)(b|X_j)) = 0$ for all $g \in E_{X_j/k}$ by the multiplicativity of the Brauer trace [Ser98, 18.1 iv]. Therefore, $(i_! a)b$ belongs to $K(Z, \mathbb{F}_\ell)_t^0$ by (4.3.2). \square

The rank function on constructible \mathbb{F}_ℓ -sheaves defines a ring homomorphism

$$\text{rk}: K(Z, \mathbb{F}_\ell) \rightarrow C(Z, \mathbb{Z}),$$

where $C(Z, \mathbb{Z})$ is the ring of constructible functions on Z with values in \mathbb{Z} . This homomorphism has a natural section which is a ring homomorphism, associating with a function $c \in C(Z, \mathbb{Z})$ the class $\langle c \rangle$ of the constructible sheaf $\oplus j_{i!} \mathbb{F}_\ell^{\text{cl}, Z_i}$, where Z is a disjoint union of locally closed subspaces $j_i: Z_i \rightarrow Z$ over which c is constant ($\langle c \rangle$ is independent of the choice of the stratification). Note that $K(Z, \mathbb{F}_\ell)_t^0$ is contained in $\text{Ker}(\text{rk})$.

Definition 4.4. For an algebraic space Z separated and of finite type over k and $a \in K(Z, \mathbb{F}_\ell)$ we will say that a is *virtually tame* if $a - \langle \text{rk}(a) \rangle$ belongs to $K(Z, \mathbb{F}_\ell)_t^0$.

We denote by $K(Z, \mathbb{F}_\ell)_t$ the subgroup of $K(Z, \mathbb{F}_\ell)$ consisting of virtually tame elements. As a subgroup, it is generated by $K(Z, \mathbb{F}_\ell)_t^0$ and the image of $\langle - \rangle$. It follows that $K(Z, \mathbb{F}_\ell)_t$ is a subring of $K(Z, \mathbb{F}_\ell)$. The rank function induces a ring isomorphism

$$K(Z, \mathbb{F}_\ell)_t / K(Z, \mathbb{F}_\ell)_t^0 \xrightarrow{\sim} C(Z, \mathbb{F}_\ell).$$

Remark 4.5. When Z is a normal connected scheme, it follows from the consequence of the valuative criterion of Gabber mentioned at the end of 4.2 that, for $a \in K_{\text{lis}}(Z, \mathbb{F}_\ell)$, a is virtually tame if and only if there exists a normal compactification \overline{Z} of Z such that, for all $g \in E_{Z/k, \overline{Z}}$, $\text{Tr}^{\text{Br}}(g, a) = \text{rk}(a)$.

The following is an immediate consequence of 4.3.

Proposition 4.6. *Let $f: Z \rightarrow W$ be a morphism of algebraic spaces of finite type over k . Then*

- (a) *The map $f^*: K(W, \mathbb{F}_\ell) \rightarrow K(Z, \mathbb{F}_\ell)$ sends $K(W, \mathbb{F}_\ell)_t$ to $K(Z, \mathbb{F}_\ell)_t$.*
- (b) *If $Z = \coprod_I Z_i$ is a partition of Z into locally closed algebraic subspaces, then $\phi: a \mapsto (a|_{Z_i})_I$ defines an isomorphism $K(Z, \mathbb{F}_\ell)_t \simeq \bigoplus_I K(Z_i, \mathbb{F}_\ell)_t$.*
- (c) *If $a, b \in K(Z, \mathbb{F}_\ell)$ such that a and ab are virtually tame, and $\text{rk}(a)$ is invertible, then b is virtually tame.*

Proposition 4.7. *Let Y be a normal, connected scheme, separated and of finite type over k , with a generic geometric point y , and let \mathcal{F} be a lisse \mathbb{F}_ℓ -sheaf on Y . Denote by $\rho: \pi_1(Y, y) \rightarrow \text{Aut}(\mathcal{F}_y)$ the representation defined by \mathcal{F} . The following conditions are equivalent:*

- (a) *$[\mathcal{F}]$ is virtually tame (4.4).*
- (b) *There exists a normal compactification \overline{Y} of Y over k such that, for all $g \in E_{Y/k, \overline{Y}}$, $\rho(g) = 1$.*

Proof. The implication (b) \Rightarrow (a) is trivial. Conversely, by 4.5, (a) implies the existence of a normal compactification \overline{Y} of Y over k such that for all $g \in E_{Y/k, \overline{Y}}$, $\text{Tr}^{\text{Br}}(g, a) = \text{rk}(a)$. The representation ρ factors through $\rho': G \rightarrow \text{Aut}(\mathcal{F}_y)$ for some finite quotient G of $\pi_1(Y, y)$. As in [Vid04, 2.1.1], let $E_{Y/k, \overline{Y}}(G)$ be the image of $E_{Y/k, \overline{Y}}$ in G . By [Ser98, 18.2, Cor. 1], the restriction of ρ' to any subgroup of G contained in $E_{Y/k, \overline{Y}}(G)$ is the trivial representation. Thus (b) follows from the fact that $E_{Y/k, \overline{Y}}(G)$ is a union of p -subgroups of G . \square

In particular:

Corollary 4.8. *Let $f: X \rightarrow Y$ be as in 4.1. The following conditions are equivalent:*

- (a) *$[f_* \mathbb{F}_\ell]$ is virtually tame (4.4).*
- (b) *X is tamely ramified at infinity over Y/k (4.1).*

4.9. Let X be an algebraic k -space of finite type, endowed with an action of a finite group G . Assume that the inertia $I(G, X)$ is finite over X and let $f: X \rightarrow Y = X/G$ be the projection. Here Y is an algebraic space of finite type over k . We say that G acts *virtually tamely* on X if $[f_*\mathbb{F}_\ell]$ is virtually tame (4.4).

To give a more concrete characterization of this property, we assume for simplicity that X is separated. We adopt the notations of 3.13. For each $S \in \mathcal{S}$, write Y_S as a finite disjoint union of normal, connected, locally closed subschemes $(Y_S)_i$, $i \in J_S$. Let $(f_S)_i: (X_S)_i \rightarrow (Y_S)_i$ be the base change of $f: X \rightarrow Y$ to $(Y_S)_i$. This is a disjoint sum of étale Galois covers $(X_H)_i$ ($H \in S$) of $(Y_S)_i$ of group $N_G(H)/H$, transitively permuted by G .

Proposition 4.10. *Using the above notations, G acts virtually tamely on X if and only if, for all $S \in \mathcal{S}$, $H \in S$ and $i \in J_S$, $(X_H)_i$ is tamely ramified at infinity over $(Y_S)_i$.*

In particular, the condition that G acts virtually tamely does not depend on ℓ .

Proof. As $f_*\mathbb{F}_\ell|(Y_S)_i = (f_S)_{i*}\mathbb{F}_\ell$, $[f_*\mathbb{F}_\ell]$ is virtually tame if and only if $[(f_S)_{i*}\mathbb{F}_\ell]$ is virtually tame for all $S \in \mathcal{S}$ and $i \in J_S$ by 4.6 (b). For all $H \in S$,

$$[(f_S)_{i*}\mathbb{F}_\ell] = (G : N_G(H))[(f_H)_{i*}\mathbb{F}_\ell],$$

where $(f_H)_i: (X_H)_i \rightarrow (Y_S)_i$ is the restriction of $(f_S)_i$. By 4.6 (c), it follows that $[(f_S)_{i*}\mathbb{F}_\ell]$ is virtually tame if and only if $[(f_H)_{i*}\mathbb{F}_\ell]$ is virtually tame. We then apply 4.8 to $(f_H)_i$. \square

The following is a generalization of 3.3, which is an analogue of Verdier's formula [Ver73, Th., p. 443].

Theorem 4.11. *Let X be an algebraic k -space, separated and of finite type, endowed with a virtually tame action of a finite group G . Then, with the notations of 4.9, we have, in $R_{\mathbb{Q}_\ell}(G)$,*

$$(4.11.1) \quad \chi(X, G, \mathbb{Q}_\ell) = \sum_{S \in \mathcal{S}} \chi(X_S/G) I_S,$$

where $I_S = [\mathbb{Q}_\ell[G/H]] \in R_{\mathbb{Q}_\ell}(G)$ for $H \in S$.

By the comparison between étale and Betti cohomologies (see the remark following 3.5), we recover Verdier's formula [Ver73, Th., p. 443] (a generalization of 3.5):

Corollary 4.12. *Let X be an algebraic space separated and of finite type over \mathbb{C} endowed with an action of a finite group G . Then, with the above notations, we have, in $R_{\mathbb{Q}}(G)$,*

$$(4.12.1) \quad \chi(X, G, \mathbb{Q}) = \sum_{S \in \mathcal{S}} \chi(X_S/G, \mathbb{Q}) I_S,$$

where $I_S = [\mathbb{Q}[G/H]] \in R_{\mathbb{Q}}(G)$ for $H \in S$.

Proof of 4.11. By the equality between χ and χ_c and the additivity of χ_c ,

$$\chi(X, G, \mathbb{Q}_\ell) = \sum_{S \in \mathcal{S}} \chi(X_S, G, \mathbb{Q}_\ell).$$

Thus we may assume $X = X_S$ for some $S \in \mathcal{S}$. By the additivity of χ again, we may assume X/G is a normal connected scheme. For $H \in S$, by 4.10, X_H is tamely ramified at infinity over X/G . Then (4.1.1) gives

$$\chi(X, N_G(H), \mathbb{Q}_\ell) = \chi(X/G)[\mathbb{Q}_\ell[N_G(H)/H]].$$

As $X = X_H \wedge^{N_G(H)} G$, we have

$$\chi(X, G, \mathbb{Q}_\ell) = \chi(X/G)[\mathbb{Q}_\ell[G/H]].$$

□

The following application of 4.11 is a generalization of a result of Petrie-Randall [PR86, 3.1] and of 3.4:

Corollary 4.13. *Under the assumptions of 4.11, suppose that G is cyclic, generated by g . Then, with the notations of 3.1,*

$$t(g) = \chi(X^G).$$

Indeed, by (4.11.1) we have

$$t(g) = \sum_{S \in \mathcal{S}} \chi(X_S/G) \operatorname{Tr}(g, I_S).$$

Now, $\operatorname{Tr}(g, I_S) = 0$ unless $S = \{G\}$, in which case $X_S = X^G$ and $\operatorname{Tr}(g, I_S) = 1$.

5 The case of rigid cohomology

The results of this section will not be used in the sequel.

5.1. We use the notations of 1.1. We assume $p > 1$ and k algebraically closed. Following [LS07, 8.2.4], we define a *realization* of X to be a sequence of immersions $X \xrightarrow{j} \overline{X} \hookrightarrow P$, where j is an open immersion, \overline{X} is a proper k -scheme and P is a formal scheme over $W = W(k)$, smooth in a neighborhood of X . We say X is *realizable* if such a realization exists. This is the case if X is quasi-projective. Indeed, if $X \subset \mathbb{P}_k^n$, we can take \overline{X} to be the closure of X in \mathbb{P}_k^n and $P = \widehat{\mathbb{P}_W^n}$. Given a realization $X \hookrightarrow \overline{X} \hookrightarrow P$, we can construct G -equivariant immersions

$$X \hookrightarrow X' = \prod_{g \in G} \overline{X} \hookrightarrow \prod_{g \in G} P$$

as in [Zhe09, 3.6]. Here G acts on the products by permutation of the factors. We obtain a G -equivariant realization by taking the closure of X in X' .

Let K be the fraction field of W . For X realizable, we denote by $H_{c, \text{rig}}^*$ the rigid cohomology with compact support of X/K in the sense of Berthelot ([Ber86, 3.1], [LS07, 8.2.5]). The action of G on X defines, by functoriality, an action of G on $H_{c, \text{rig}}^*(X/K)$. As the category of $K[G]$ -modules is semisimple, this action turns the complex $R\Gamma_{c, \text{rig}}(X/K)$ defining $H_{c, \text{rig}}^*(X/K)$ into an object of $D^b(K[G])$. Such a complex can also be defined directly as follows. Choose a G -equivariant realization $X \hookrightarrow \overline{X} \hookrightarrow P$. Then we have

$$R\Gamma_{c, \text{rig}}(X/K) = R\Gamma(\overline{X}[P, R\Gamma_{\overline{X}[P]}(\Omega_{\overline{X}[P]}^\bullet)).$$

By Berthelot's finiteness theorem [Ber97, 3.9 (i)], this complex has finite-dimensional cohomology groups, and, when X/k is *proper and smooth*, is isomorphic to $R\Gamma(X/W) \otimes_W^L K$, where $R\Gamma(X/W) \in D^b(W[G])$ is the complex calculating the crystalline cohomology of X/W [Ber97, 1.9]. Recall that, for any proper and smooth k -scheme X , $R\Gamma(X/W)$ can be computed by $R\Gamma(X, W\Omega_X^\bullet)$, where $W\Omega_X^\bullet$ is the de Rham-Witt complex of X/k [Ill79, I 1.15, II (2.8.2)], which is a complex of G - W -modules on (the Zariski site of) X . Note that here the category of $W[G]$ -modules is no longer semisimple, and this complex can't be recovered from the mere datum of its cohomology groups, the $W[G]$ -modules $H^i(X/W)$.

One can't expect in general that, if G acts freely, $R\Gamma_{c,\text{rig}}(X/K)$ comes by extension of scalars from a perfect complex of $W[G]$ -modules. Indeed, if it were the case, the traces of p -singular elements would be zero [Ser98, 16.2, Th. 36], and in the example given after (1.10.1), the trace of s on $H_{c,\text{rig}}^*(X/K)$ can be shown to be equal to 1. We have, however, the following results, which complement 3.3:

Theorem 5.2. *Let X/k with the action of G be as in 1.2, with $p > 1$. With the notations of 5.1:*

- (a) *If X is realizable, $\text{Tr}(s, R\Gamma_{c,\text{rig}}(X/K)) := \sum (-1)^i \text{Tr}(s, H_{c,\text{rig}}^i(X/K))$ is equal to the integer $t(s)$ of 3.2 (i), for all $s \in G$.*
- (b) *If X/k is proper and smooth, and G acts freely on X , then $R\Gamma(X/W)$ is a perfect complex of $W[G]$ -modules.*

For the proof we need the following well known lemmas:

Lemma 5.3. *Let X/k be a projective and smooth scheme, s a k -endomorphism of X , ℓ a prime $\neq p$. Then we have an equality of rational integers:*

$$(5.3.1) \quad \text{Tr}(s, H^*(X/W) \otimes K) = \text{Tr}(s, H^*(X, \mathbb{Q}_\ell)).$$

Proof. Indeed, if $\text{CH}^*(X)$ is the Chow ring of X , the theory of cycle classes in ℓ -adic (resp. crystalline) cohomology ([SGA 4 $\frac{1}{2}$, Cycle], [Lau76]) (resp. [Gro85]) gives a homomorphism $\text{CH}^*(X) \rightarrow H^*(X, \mathbb{Q}_\ell)$ (resp. $\text{CH}^*(X) \rightarrow H^*(X/W) \otimes K$), which is multiplicative and compatible with Gysin maps. This implies that both sides of (5.3.1) are equal to the intersection number of the graph of s and the diagonal in $X \times X$. \square

Lemma 5.4. *Let $f: X \rightarrow Y$ be a finite universal homeomorphism between k -schemes (resp. realizable k -schemes) separated of finite type. Then the canonical homomorphism*

$$R\Gamma_{\text{rig}}(Y/K) \rightarrow R\Gamma_{\text{rig}}(X/K), \quad (\text{resp. } R\Gamma_{c,\text{rig}}(Y/K) \rightarrow R\Gamma_{c,\text{rig}}(X/K))$$

is an isomorphism.

Proof. If f is a bijective immersion, the assertions follow from the definitions. In the general case, the diagonal morphism $X \rightarrow \text{cosk}_0(X/Y)_n$ is a bijective immersion. Thus the assertion for $R\Gamma_{\text{rig}}$ follows from cohomological descent for finite morphisms [Tsu03, 4.5.1]. For $R\Gamma_{\text{rig},c}$, we proceed by induction on $\dim Y$. For any closed subscheme V of Y , we have a distinguished triangle [Ber86, 3.1]

$$R\Gamma_{c,\text{rig}}((Y - V)/K) \rightarrow R\Gamma_{c,\text{rig}}(Y/K) \rightarrow R\Gamma_{c,\text{rig}}(X/K) \rightarrow .$$

Replacing Y by Y_{red} and shrinking Y , we may thus assume Y quasi-projective, normal and integral. Let \overline{Y} be a normal projective compactification of Y . Factorize $X \rightarrow \overline{Y}$ into a dense open immersion and a finite morphism:

$$X \hookrightarrow \overline{X} \xrightarrow{\overline{f}} \overline{Y}.$$

Then \overline{f} is a universal homeomorphism. We are thus reduced to the case Y projective, where the assertions for $R\Gamma_{c,\text{rig}}$ and for $R\Gamma_{\text{rig}}$ coincide. \square

Lemma 5.5. *Let $a < b$ be integers, $C \in D^{[a,b]}(W[G])$. Assume that the tor-amplitude of $C \otimes_W^L k \in D(k[G])$ is contained in $[a+1, b]$. Then the tor-amplitude of C is contained in $[a, b]$.*

Proof. The short exact sequence

$$(5.5.1) \quad 0 \rightarrow W \xrightarrow{\times p} W \rightarrow k \rightarrow 0$$

is resolution of k by free W -modules. Let M be a right $W[G]$ -module. Then $M \otimes_W^L k \in D^{[-1,0]}$. Moreover $- \otimes_{K[G]} -$ is an exact bifunctor. Thus

$$\begin{aligned} (M \otimes_{W[G]}^L C) \otimes_W^L k &\simeq (M \otimes_W^L k) \otimes_{k[G]}^L (C \otimes_W^L k) \in D^{[a,b]}(k[G]), \\ (M \otimes_{W[G]}^L C) \otimes_W K &\simeq (M \otimes_W K) \otimes_{K[G]} (C \otimes_W K) \in D^{[a,b]}(K[G]). \end{aligned}$$

Putting $E = M \otimes_{W[G]}^L C$ for brevity, (5.5.1) induces the long exact sequence

$$H^{q-1}(E \otimes_W^L k) \rightarrow H^q(E) \xrightarrow{\times p} H^q(E) \rightarrow H^q(E \otimes_W^L k).$$

For $q < a$, $H^q(E) \xrightarrow{\times p} H^q(E)$ is thus an isomorphism. Moreover, $H^q(E) \otimes_W K = 0$. Thus $H^q(E) = 0$. \square

We prove 5.2 (a) by induction on the dimension d of X , using de Jong's Galois alterations, as in [Vid04, 4.4] and [Zhe09, § 3]. The assertion is trivial for $d = 0$. Assume $d \geq 1$. By 5.4, we may assume X reduced. Using the inductive hypothesis and the additivity of traces on $H_{c,\text{rig}}^*$, we may replace X by a dense open G -invariant subscheme. Therefore we may assume X smooth, affine. The connected components of X are permuted by s , and the trace of s is the sum of the traces of s on the cohomology of those components which are stabilized by s . So we may assume furthermore X integral. Choose a G -equivariant dense open embedding $j: X \rightarrow Z$, with Z/k a projective, integral G -scheme. By Gabber's refinement of de Jong's results on equivariant alterations [Zhe09, 3.8] there exist the following data:

- a surjective homomorphism $u: G' \rightarrow G$ of finite groups,
- a projective smooth, integral k -scheme Z' endowed with an action of G' , and a surjective u -equivariant k -morphism $a: Z' \rightarrow Z$,
- a G -stable dense open affine subscheme V of X ,

satisfying the following property:

- if H is the kernel of $u: G' \rightarrow G$, and $V' = a^{-1}(V)$, $a|V: V' \rightarrow V$ factors into

$$V' \rightarrow V'/H \rightarrow V,$$

where $V' \rightarrow V'/H$ is an étale Galois cover of group H/H_0 , with $H_0 = H \cap \text{Ker}(G' \rightarrow \text{Aut}(k(\eta')))$ (η' the generic point of X'), and $V'/H \rightarrow V$ is a finite and flat universal homeomorphism.

The morphism a is sometimes called a *Galois alteration*. One may further assume that $Z' - a^{-1}(X)$ is contained in a strict normal crossing divisor of Z' . We don't need this more precise form.

By the inductive assumption, it suffices to show the assertion for (V, G) . By 5.4, we may replace (V, G) by (U, G) , where $U = V'/H$. We have

$$\begin{aligned} \text{Tr}(s, H_c^*(U, \mathbb{Q}_\ell)) &= \frac{1}{(H : H_0)} \sum \text{Tr}(s', H_c^*(V', \mathbb{Q}_\ell)), \\ \text{Tr}(s, H_{c,\text{rig}}^*(U/K)) &= \frac{1}{(H : H_0)} \sum \text{Tr}(s', H_{c,\text{rig}}^*(V'/K)), \end{aligned}$$

the sums being extended to the classes modulo H_0 of elements $s' \in G'$ above s . We may therefore replace (U, G) by (V', G') . By the inductive assumption, we may replace (V', G') by (Z', G') , and we conclude by 5.3.

By 5.5, to prove 5.2 (b), it is enough to show that $R\Gamma(X/W) \otimes_W^L k$ is a perfect complex of $k[G]$ -modules. We have, by [BO78, 7.1, 7.24],

$$R\Gamma(X/W) \otimes_W^L k = R\Gamma(X/k) = R\Gamma(X, \Omega_{X/k}^\bullet) = R\Gamma(Y, \pi_* \Omega_{X/k}^\bullet)$$

in $D^b(k[G])$, where $Y = X/G$ and $\pi: X \rightarrow Y$ is the projection. As π is an étale Galois cover of group G , $\pi_* \mathcal{O}_X$ is, étale locally on Y , isomorphic to $\mathcal{O}_Y[G]$, in particular, is flat over $k[G]$. The same is true of $\pi_* \Omega_{X/k}^i = \pi_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \Omega_{Y/k}^i$. For any right $k[G]$ -module M , by the projection formula,

$$M \otimes_{k[G]}^L R\Gamma(Y, \pi_* \Omega_{X/k}^i) \simeq R\Gamma(Y, M \otimes_{k[G]} (\pi_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \Omega_{Y/k}^i))$$

has cohomology concentrated in $[0, d]$, where $d = \dim(X) = \dim(Y)$. Thus $R\Gamma(Y, \pi_* \Omega_{X/k}^i)$ is a perfect complex of $k[G]$ -modules. Filtering $\pi_* \Omega_{X/k}^\bullet$ by the naïve filtration, we get

$$\text{Gr } R\Gamma(Y, \pi_* \Omega_{X/k}^\bullet) \simeq \bigoplus_{i \in \mathbb{Z}} R\Gamma(Y, \pi_* \Omega_{X/k}^i[-i]),$$

which implies that $R\Gamma(X/k)$ is perfect over $k[G]$ (even as a filtered complex [Ill71, V 3.1]).

Remark 5.6. (a) In the situation of 5.2 (a), assume X/k proper. Then, if G acts freely on X , $\text{Tr}(s, R\Gamma_{c,\text{rig}}(X/K)) = 0$ for $s \in G$, $s \neq 1$. Indeed, $t_c(s) = 0$ by the Lefschetz-Verdier trace formula [SGA 5, III]. It seems that so far no general Lefschetz-Verdier formula is available in rigid cohomology. For example, if u is a fixed point free endomorphism of X/k , we don't know whether $\text{Tr}(u, R\Gamma_{c,\text{rig}}(X/K)) = 0$. This vanishing holds at least in the smooth case (5.3).

(b) In the situation of 5.2 (a), it is unknown whether one has

$$\text{Tr}(s, H_{c,\text{rig}}^*(X/K)) = \text{Tr}(s, H_{\text{rig}}^*(X/K)),$$

even when $G = \{1\}$.

6 Fixed point sets: around a theorem of P. Smith

The results in this section were suggested to the first author by Serre. They overlap with parts of [Ser09, §§ 7, 8].

Proposition 6.1. *[Ser09, 7.2] Let k be an algebraically closed field of characteristic p , ℓ a prime number $\neq p$, and X an algebraic space separated and of finite type over k , equipped with an action of an ℓ -group G . Then:*

$$(6.1.1) \quad \chi(X^G) \equiv \chi(X) \pmod{\ell}.$$

Proof. By additivity of χ , we may assume X separated. If G is an extension $0 \rightarrow G_1 \rightarrow G \rightarrow G_2 \rightarrow 0$, X^{G_1} is G -stable, G acts on it through G_2 , and $X^G = (X^{G_1})^{G_2}$. So by induction we may assume that G is cyclic of order ℓ . Then G acts freely on $U := X - X^G$, hence, by 3.3, $\chi(U) = \ell\chi(U/G)$, and (6.1.1) follows by additivity of χ . One could also deduce (6.1.1) from 4.13: $t(g) = \chi(X^G)$ for all $g \neq 1$, and $\sum_{g \in G} t(g) \equiv 0 \pmod{\ell}$. \square

As Serre observes in (*loc. cit.*), 6.1 implies that if $\chi(X)$ is not divisible by ℓ , then X^G is not empty. If X is the affine space \mathbb{A}_k^n , we find $\chi(X^G) \equiv 1 \pmod{\ell}$. In this case, Smith's theory, as recalled in (*loc. cit.*) gives more.

6.2. Let k be an algebraically closed field of characteristic p , X an algebraic space separated and of finite type over k , and let ℓ be a prime number, possibly equal to p . We say that X is mod ℓ *acyclic* if $H^i(X, \mathbb{F}_\ell) = 0$ for $i \neq 0$ and $H^0(X, \mathbb{F}_\ell) = \mathbb{F}_\ell$. When ℓ is different from p , by the finiteness of $H^*(X, \mathbb{Z}_\ell)$ and the exact sequence of universal coefficients, X is mod ℓ acyclic if and only if $H^i(X, \mathbb{Z}_\ell) = 0$ for $i \neq 0$ and $H^0(X, \mathbb{Z}_\ell) = \mathbb{Z}_\ell$. When $\ell = p$, if X/k is proper, connected, and $H^i(X, \mathcal{O}) = 0$ for $i > 0$, then, by the Artin-Schreier exact sequence, X is mod ℓ acyclic. The following (for schemes) is [Ser09, 7.5 b)]. This result and 6.9 below (for schemes) were obtained independently by Morin [Mor08, Th. 2.46], assuming $G = \mathbb{F}_\ell$, $\ell \neq p$ and X^G contained in an affine open subset. His method is similar to ours and is based on a variant à la Tate of equivariant cohomology.

Theorem 6.3. *Let X/k and ℓ be as in 6.2. Assume that an ℓ -group G acts on X/k and that X is mod ℓ acyclic. Then so is X^G .*

Here is a proof using equivariant cohomology, as in [Bor55]. By dévissage, as in the proof of 6.1, we may assume that G is a cyclic group of order ℓ . If S/k is an algebraic space acted on by G and F a G - \mathbb{F}_ℓ -sheaf on S , we denote by $R\Gamma_G(S, F)$ the complex $R\Gamma([S/G], F)$ (where $[S/G]$ is the associated Deligne-Mumford stack with its étale topology), which can be calculated as $R\Gamma(S_\bullet, F_\bullet)$, where $S_\bullet = \text{cosk}_0(S/[S/G])$ is the simplicial algebraic space defined by the action of G on S and F_\bullet the corresponding simplicial sheaf on S_\bullet . We have $R\Gamma(S, F) \in D^+(\mathbb{F}_\ell[G])$ and $R\Gamma_G(S, F) = R\Gamma(G, R\Gamma(S, F))$. When F is the constant sheaf \mathbb{F}_ℓ , we write $R\Gamma_G(S)$ for $R\Gamma_G(S, F)$. The projection $S \rightarrow \text{Spec } k$ makes $H_G^*(S) = \bigoplus_{i \geq 0} H_G^i(S)$ into a graded algebra over the graded \mathbb{F}_ℓ -algebra

$$R = H_G^*(\text{Spec } k) = H^*(G, \mathbb{F}_\ell),$$

and $H_G^*(S, F)$ into a graded module over R .

6.3.1. Recall that when $\ell = 2$, R is a polynomial algebra $\mathbb{F}_\ell[x]$ in one generator of degree 1, and when $\ell > 2$, R is the graded tensor product of the algebra of dual numbers $\mathbb{F}_\ell[x]/(x^2)$ with x of degree 1 by a polynomial algebra $\mathbb{F}_\ell[y]$ with y of degree 2 [CE56, XII 7].

Let $Y = X^G$, $U = X - Y$, $u: U \rightarrow X$ the inclusion. The (equivariant) short exact sequence $0 \rightarrow u_! \mathbb{F}_{\ell U} \rightarrow \mathbb{F}_{\ell X} \rightarrow \mathbb{F}_{\ell Y} \rightarrow 0$ gives a long exact sequence of equivariant cohomology

$$(6.3.2) \quad \cdots \rightarrow H_G^*(X, u_! \mathbb{F}_\ell) \rightarrow H_G^*(X) \rightarrow H_G^*(Y) \rightarrow H_G^{*+1}(X, u_! \mathbb{F}_\ell) \rightarrow \cdots,$$

where $H_G^* = \bigoplus_i H_G^i$. This is an exact sequence of graded R -modules. Consider the commutative diagram (3.13.1). As G is cyclic of order ℓ , G acts freely on U , so $(f_U)_* \mathbb{F}_{\ell U}$ is locally free of rank one over $\mathbb{F}_\ell[G]$, hence $R\Gamma(G, v_!(f_U)_* \mathbb{F}_{\ell U}) = v_!(\mathbb{F}_\ell)_{U/G}$. Therefore

$$(6.3.3) \quad \begin{aligned} R\Gamma(G, R\Gamma(X, u_! \mathbb{F}_\ell)) &= R\Gamma(G, R\Gamma(X/G, f_* u_! \mathbb{F}_\ell)) = R\Gamma(X/G, R\Gamma(G, f_* u_! \mathbb{F}_\ell)) \\ &= R\Gamma(X/G, R\Gamma(G, v_!(f_U)_* \mathbb{F}_\ell)) = R\Gamma(X/G, v_! \mathbb{F}_\ell), \end{aligned}$$

so that

$$H_G^*(X, u_! \mathbb{F}_\ell) = H^*(X/G, v_! \mathbb{F}_\ell)$$

is a graded module of bounded degree, as $\text{cd}_\ell(X/G)$ is finite by Lemma 6.4 below (whether ℓ is different from p or not). As $R\Gamma(X, \mathbb{F}_\ell) = \mathbb{F}_\ell$, we have $H_G^*(X) = R$. Therefore, in (6.3.2) the map $H_G^*(X, u_! \mathbb{F}_{\ell U}) \rightarrow H_G^*(X)$ vanishes, and (6.3.2) boils down to a short exact sequence

$$0 \rightarrow H_G^*(X) \rightarrow H_G^*(Y) \rightarrow H^{*+1}(X/G, v_! \mathbb{F}_\ell) \rightarrow 0,$$

which can be rewritten

$$(6.3.4) \quad 0 \rightarrow R \rightarrow R \otimes_{\mathbb{F}_\ell} H^*(Y) \rightarrow H^{*+1}(X/G, v_! \mathbb{F}_\ell) \rightarrow 0,$$

since, by Künneth's formula for $BG \times Y$,

$$(6.3.5) \quad H_G^*(Y) = H_G^* \otimes_{\mathbb{F}_\ell} H^*(Y) = R \otimes_{\mathbb{F}_\ell} H^*(Y).$$

By Lemma 6.5 below, this implies that $H^*(Y)$ is free of rank 1 over \mathbb{F}_ℓ , hence Y is mod ℓ acyclic.

Lemma 6.4. *Let X be a separated algebraic space of dimension d of finite type over k . Then the ℓ -cohomological dimension $\text{cd}_\ell X$ is at most $2d$ (resp. d) if $\ell \neq p$ (resp. $\ell = p$).*

We prove this lemma by induction on d . By Chow's lemma [Knu71, IV.3.1], we can choose

$$\begin{array}{ccc} & & X' \\ & \nearrow j' & \downarrow \pi \\ U & \xrightarrow{j} & X \end{array}$$

where X' is a scheme, π is proper and is an isomorphism over a dense open subscheme U of X . Let \mathcal{F} be an ℓ -torsion sheaf on X . Considering the short exact sequence

$$0 \rightarrow j_! j^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow Q \rightarrow 0$$

and applying the induction hypothesis to Q , it is enough to show that for any ℓ -torsion sheaf \mathcal{G} on U , we have

$$H^i(X, j_! \mathcal{G}) = 0, \quad i > 2d \text{ (resp. } i > d \text{)}.$$

Since π is proper, we have $j_! \mathcal{G} \simeq R\pi_* j'_! \mathcal{G}$, so we have $H^i(X, j_! \mathcal{G}) = H^i(X', j'_! \mathcal{G})$. Hence the lemma follows from the scheme case, which is well-known [SGA 4, X 4.3] (resp. [SGA 4, X 5.2]).

Lemma 6.5. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence of graded R -modules, with L and M free, and N of bounded degree. Then L and M have the same rank over R .*

Indeed, if $\ell = 2$ (resp. $\ell > 2$), $N \otimes_R R[x^{-1}] = 0$ (resp. $N \otimes_R R[y^{-1}] = 0$).

Remark 6.6. For X and ℓ as in 6.2, X is mod ℓ acyclic if and only if

$$\sum_i \dim H^i(X, \mathbb{F}_\ell) = 1.$$

One can consider the analogous condition

$$(6.6.1) \quad \sum_i \dim H_c^i(X, \mathbb{F}_\ell) = 1.$$

If $\ell \neq p$, (6.6.1) is equivalent to $R\Gamma_c(X, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell[-2d]$ and to $R\Gamma_c(X, \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell[-2d]$, which imply that X is irreducible. Here $d = \dim X$. If $\ell \neq p$ and X is smooth over k , then (6.6.1) is equivalent to X being mod ℓ acyclic by Poincaré duality.

For an arbitrary ℓ , assume that an ℓ -group G acts on X . Using arguments similar to the proof of 6.3, one can show that (6.6.1) implies $\sum_i \dim H_c^i(X^G, \mathbb{F}_\ell) = 1$. The case $\ell \neq p$ was obtained by Symonds [Sym04, 4.3], using a different method based on the theorem of Rickard mentioned in 3.12 (c).

Corollary 6.7. *Let X and ℓ be as in 6.2. Assume that X is mod ℓ acyclic, and that X is endowed with an action of a finite group G by k -automorphisms. Then X/G is mod ℓ acyclic.*

Proof. Let $f: X \rightarrow Y = X/G$ be the projection. Let us first show

$$(6.7.1) \quad (f_* \mathbb{F}_\ell)^G \simeq \mathbb{F}_\ell.$$

Let $Y' \rightarrow Y$ be an étale morphism of finite type. We have $(f_* \mathbb{F}_\ell)(Y) \simeq \mathbb{F}_\ell^{\pi_0(X')}$, where $X' = X \times_Y Y'$. Since the projection $X' \rightarrow Y'$ identifies Y' with the quotient of X' by G ,

$$(f_* \mathbb{F}_\ell)^G(Y') \simeq \mathbb{F}_\ell^{\pi_0(X')/G} \simeq \mathbb{F}_\ell^{\pi_0(Y')} \simeq \mathbb{F}_\ell(Y').$$

(a) Case where G is an ℓ -group. If H is a normal subgroup of G , G acts on X/H through G/H and $X/G = (X/H)/(G/H)$. Therefore we may assume G cyclic of order ℓ . In this case, by 6.3 the restriction map $R\Gamma(X, \mathbb{F}_\ell) \rightarrow R\Gamma(X^G, \mathbb{F}_\ell)$ is an isomorphism, hence $R\Gamma(X, u_! \mathbb{F}_\ell) = 0$ with the notations of (6.3.2). Therefore, by (6.3.3) $R\Gamma(X/G, v_! \mathbb{F}_\ell) = 0$, hence the restriction map $R\Gamma(X/G, \mathbb{F}_\ell) \rightarrow R\Gamma(X^G, \mathbb{F}_\ell)$ is an isomorphism. Finally, by 6.3, $R\Gamma(X^G, \mathbb{F}_\ell) = \mathbb{F}_\ell$.

(b) General case. Let L be an ℓ -Sylow subgroup of G . In order to relate $H^*(X/G, \mathbb{F}_\ell)$ to $H^*(X/L, \mathbb{F}_\ell)$, we consider the commutative square

$$\begin{array}{ccc} X \wedge^L G & \xrightarrow{h} & X \wedge^G G \\ \downarrow f' & & \downarrow f \\ X \wedge^L \{1\} & \xrightarrow{g} & X \wedge^G \{1\} \end{array}$$

See 3.13 for the definition of the contracted products. Since h is a finite étale cover of fibers isomorphic to G/L , hence of degree $d = (G : L)$ prime to ℓ , the composition

$$(\mathbb{F}_\ell)_X \rightarrow h_*(\mathbb{F}_\ell)_{X \wedge^L G} \rightarrow (\mathbb{F}_\ell)_X$$

of the adjunction map and the trace map is multiplication by d [SGA 4, IX 5.1.4]. Applying f_* and taking G -invariants, we see that the composition

$$(f_*(\mathbb{F}_\ell)_X)^G \rightarrow (f_*h_*(\mathbb{F}_\ell)_{X \wedge^L G})^G \rightarrow (f_*(\mathbb{F}_\ell)_X)^G$$

is again multiplication by d . Hence $(\mathbb{F}_\ell)_{X/G} \simeq (f_*(\mathbb{F}_\ell)_X)^G$ (6.7.1) is a direct factor of

$$g_*(\mathbb{F}_\ell)_{X/L} \simeq g_*(f'_*(\mathbb{F}_\ell)_{X \wedge^L G})^G \simeq (f_*h_*(\mathbb{F}_\ell)_{X \wedge^L G})^G.$$

It follows that g induces an injection $H^*(X/G, \mathbb{F}_\ell) \rightarrow H^*(X/L, \mathbb{F}_\ell)$ and therefore an isomorphism $H^*(X/G, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell$ by case (a) above. \square

There are many variants and generalizations of 6.3. Here are two of them.

Theorem 6.8. (cf. [Ser09, 7.5 a]) *Let X/k and ℓ be as in 6.2. Assume that an ℓ -group G acts on X/k . Let N be an integer such that $H^i(X, \mathbb{F}_\ell) = 0$ for $i > N$. Then $H^i(X^G, \mathbb{F}_\ell) = 0$ for $i > N$.*

We may assume G cyclic of order ℓ . Serre's proof of 6.3 and 6.8 (*loc. cit.*) makes no use of equivariant cohomology, but instead exploits the action of $\mathbb{F}_\ell[G]$ on $f_*\mathbb{F}_\ell$, with the notations above. It is also easy to prove 6.8 along the lines of the above proof of 6.3. Again, the key point is that, by 6.4 applied to $Y = X^G$, the restriction homomorphism

$$r: H_G^*(X) \rightarrow H_G^*(Y) = R \otimes_{\mathbb{F}_\ell} H^*(Y)$$

(6.3.5) is a TN-isomorphism of graded R -modules (in the sense of [EGA II, 3.4]), i.e. there exists an integer n_0 such that $r_n: H_G^n(X) \rightarrow H_G^n(Y)$ is an isomorphism for $n \geq n_0$. The source and target of r are the abutments of spectral sequences

$$\begin{aligned} E(X): E_2^{ij} &= H^i(G, H^j(X)) \Rightarrow H_G^{i+j}(X), \\ E(Y): E_2^{ij} &= H^i(G, H^j(Y)) \Rightarrow H_G^{i+j}(Y), \end{aligned}$$

the second one being degenerate at E_2 , and r underlies a morphism of spectral sequences $E(X) \rightarrow E(Y)$. Take $n \geq \sup(n_0, N)$. Then $H_G^n(X) = F^{n-N}H_G^n(X)$ (where F^\bullet denotes the filtration on the abutment). As r_n is a filtered isomorphism, this implies that $H_G^n(Y) = F^{n-N}H_G^n(Y)$, and consequently that $H^j(Y) = 0$ for $N < j \leq n$.

Theorem 6.9. *Let k and ℓ be as in 6.2, G be an ℓ -group, $f: X \rightarrow X'$ be a G -equivariant morphism of algebraic k -spaces which are separated of finite type. We denote by $R\Gamma(X'/X, \mathbb{F}_\ell)$ the complex defining the relative cohomology of X' modulo X . Assume that $R\Gamma(X'/X, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell[-N]$. Then there exists $M \leq N$ such that $\ell(N - M)$ is even and $R\Gamma(X'^G/X^G, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell[-M]$.*

Relative cohomology is defined in [Del74, 6.3] and more generally in [Ill71, III 4.10]. We have an exact triangle

$$R\Gamma(X'/X, \mathbb{F}_\ell) \rightarrow R\Gamma(X', \mathbb{F}_\ell) \rightarrow R\Gamma(X, \mathbb{F}_\ell) \rightarrow .$$

Proof. We may assume that G is cyclic of order ℓ . Let $u: X - X^G \hookrightarrow X$, $u': X' - X'^G \hookrightarrow X'$. The first line of the 9-diagram

$$\begin{array}{ccccccc} C & \longrightarrow & R\Gamma(X'/X) & \longrightarrow & R\Gamma(X'^G/X^G) & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ R\Gamma(X', u'_! \mathbb{F}_\ell) & \longrightarrow & R\Gamma(X') & \longrightarrow & R\Gamma(X'^G) & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ R\Gamma(X, u_! \mathbb{F}_\ell) & \longrightarrow & R\Gamma(X) & \longrightarrow & R\Gamma(X^G) & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \end{array}$$

gives rise to a long exact sequence

$$(6.9.1) \quad \cdots \rightarrow H^*(G, C) \rightarrow H_G^*(X'/X) \rightarrow H_G^*(X'^G/X^G) \rightarrow H^{*+1}(G, C) \rightarrow \cdots ,$$

where $H_G^*(X'/X) = H^*(G, R\Gamma(X'/X))$, $H_G^*(X'^G/X^G) = H^*(G, R\Gamma(X'^G/X^G))$. By (6.3.3) and 6.4, $H^*(G, C)$ is of bounded degree. We have a spectral sequence

$$E(X'/X): E_2^{ij} = H^i(G, H^j(X'/X)) \Rightarrow H_G^{i+j}(X'/X),$$

analogous to $E(X)$, whose E_2 term is concentrated on the horizontal line of degree N , and therefore degenerates at E_2 , yielding isomorphisms $E_2^{i,N} \xrightarrow{\sim} H_G^{i+N}(X'/X)$. As G can't act on \mathbb{F}_ℓ but trivially, we get $H_G^*(X'/X) \simeq R \otimes H^N(X'/X)$, i.e. $H_G^*(X'/X) \simeq R(-N)$, where $(-)$ is the usual shift on graded modules. Then (6.9.1) boils down to an exact sequence

$$0 \rightarrow R(-N) \rightarrow R \otimes_{\mathbb{F}_\ell} H^*(X'^G/X^G) \rightarrow H^{*+1}(G, C) \rightarrow 0,$$

which shows that $R \otimes H^*(X'^G/X^G)$ is (graded) free of rank one on R , and therefore that $H^*(X'^G/X^G)$ is concentrated in one degree $M \leq N$, and of dimension one over \mathbb{F}_ℓ . If $\ell > 2$, as the graded pieces of R with odd degrees are killed by x (6.3.1), $N - M$ must be even. \square

Let Y be an algebraic subspace of X . We say the pair (X, Y) is a *mod ℓ cohomology N -disk* if $R\Gamma(X/Y, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell[-N]$.

Corollary 6.10. *Let X/k and ℓ be as in 6.2, G be an ℓ -group acting on X , Y be a G -equivariant algebraic subspace of X . Assume that (X, Y) is a mod ℓ cohomology N -disk for some $N \geq 0$. Then (X^G, Y^G) is a mod ℓ cohomology M -disk for some $0 \leq M \leq N$ with $\ell(N - M)$ even.*

Let X be an algebraic k -space separated of finite type. We write

$$R\tilde{\Gamma}(X, \mathbb{F}_\ell) = R\Gamma(\mathrm{Spec} k/X, \mathbb{F}_\ell)[1] \in D^{\geq -1}.$$

It is a cone of the adjunction morphism $\mathbb{F}_\ell = R\Gamma(\mathrm{Spec} k, \mathbb{F}_\ell) \rightarrow R\Gamma(X, \mathbb{F}_\ell)$. We say that X is a *mod ℓ cohomology N -sphere* if $R\Gamma(X, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell[-N]$. For $N = -1$ (resp. $N = 0$, resp. $N \geq 1$), X is a mod ℓ cohomology N -sphere if and only if X is empty (resp. $R\Gamma(X, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell \oplus \mathbb{F}_\ell$, resp. $H^0(X, \mathbb{F}_\ell) \simeq H^N(X, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell$ and $H^q(X, \mathbb{F}_\ell) = 0$ for $q \neq 0, N$). Applying 6.9 to the structure morphism $X \rightarrow \mathrm{Spec} k$, we obtain the following.

Corollary 6.11. *Let X/k and ℓ be as in 6.2. Assume that an ℓ -group G acts on X and that X is a mod ℓ cohomology N -sphere for some $N \geq -1$. Then X^G is a mod ℓ cohomology M -sphere with $-1 \leq M \leq N$ and $\ell(N - M)$ even.*

These corollaries are analogues of [Bre72, III 5.1, 5.2]. 6.11 is an analogue of the main result in [Bor55].

7 A localization theorem

In this section we fix a prime number ℓ and an algebraically closed field k of characteristic p .

In the situation of 6.3, with $G = \mathbb{F}_\ell$, the restriction map $H_G^*(X) \rightarrow H_G^*(X^G)$ induces an isomorphism of graded R -algebras $H_G^*(X)[y^{-1}] \xrightarrow{\sim} H_G^*(X^G)[y^{-1}]$ (resp. $H_G^*(X)[x^{-1}] \xrightarrow{\sim} H_G^*(X^G)[x^{-1}]$) for $\ell > 2$ (resp. $\ell = 2$). We give a generalization in this section.

7.1. Following Quillen's terminology [Qui71, § 4], for $r \in \mathbb{N}$, by an *elementary abelian ℓ -group of rank r* we mean a group G isomorphic to the direct product of r cyclic groups of order ℓ , i.e. the group underlying a vector space of dimension r over \mathbb{F}_ℓ (such groups are sometimes called groups of type $(\ell, \ell, \dots, \ell)$). Consider the short exact sequence

$$0 \rightarrow \mathbb{F}_\ell \rightarrow \mathbb{Z}/\ell^2\mathbb{Z} \rightarrow \mathbb{F}_\ell \rightarrow 0,$$

with trivial G actions. It induces an exact sequence

$$\mathrm{Hom}(G, \mathbb{F}_\ell) \xrightarrow{\sim} \mathrm{Hom}(G, \mathbb{Z}/\ell^2\mathbb{Z}) \rightarrow \mathrm{Hom}(G, \mathbb{F}_\ell) \xrightarrow{\beta} H^2(G, \mathbb{F}_\ell),$$

where we have identified $H^1(G, \mathbb{F}_\ell)$ with $\check{G} = \mathrm{Hom}(G, \mathbb{F}_\ell)$ by the natural isomorphism. It follows that the Bockstein operator β is injective. Recall (*loc. cit.*) that we have a natural identification of \mathbb{F}_ℓ -graded algebras

$$H^*(G, \mathbb{F}_\ell) = \begin{cases} S(\check{G}) & \text{if } \ell = 2 \\ \Lambda(\check{G}) \otimes S(\beta\check{G}) & \text{if } \ell > 2 \end{cases}$$

where S (resp. Λ) denotes a symmetric (resp. exterior) algebra over \mathbb{F}_ℓ . In particular, if $\{x_1, \dots, x_r\}$ is a basis of \check{G} over \mathbb{F}_ℓ , then

$$H^*(G, \mathbb{F}_\ell) = \begin{cases} \mathbb{F}_\ell[x_1, \dots, x_r] & \text{if } \ell = 2 \\ \Lambda(x_1, \dots, x_r) \otimes \mathbb{F}_\ell[y_1, \dots, y_r] & \text{if } \ell > 2 \end{cases}$$

where $y_i = \beta x_i$.

We put $V_G = \text{Spec}(S(\beta\check{G}))$. It is an affine space of dimension r over \mathbb{F}_ℓ . For any subgroup H of G , the surjection $\check{G} \rightarrow \check{H}$ induces a closed immersion $V_H \hookrightarrow V_G$. For $f \in S(\beta\check{G})$, we denote by $V_G(f)$ the closed subset of V_G defined by f . Then

$$V_H = \bigcap_{x \in \text{Ker}(\check{G} \rightarrow \check{H})} V_G(\beta x).$$

For any $S(\beta\check{G})$ -module M , we denote by $\text{Supp}_G(M) \subset V_G$ the support of M and by $\overline{\text{Supp}}_G(M)$ the Zariski closure of $\text{Supp}_G(M)$.

The following is an analogue of the localization theorem of Borel-Atiyah-Segal for actions of tori [GKM98, 6.2]. As in 3.13, for any algebraic space X separated of finite type over k , endowed with an action of G , and any subgroup H of G , we put $X_H = X^H - \bigcup_{H'} X^{H'}$, where H' runs over subgroups of G strictly containing H . Since G is abelian, G acts on X^H and X_H . As in 6.2 we do not assume ℓ invertible in k .

Theorem 7.2. *Let G be an elementary abelian ℓ -group, $Y \rightarrow X$ be a G -equivariant closed immersion between algebraic spaces separated and of finite type over k . Let $j: U = X - Y \hookrightarrow X$ be the complementary open immersion, \mathcal{H} be the set of subgroups H of G such that $R\Gamma(X/G, v_{H!}\mathbb{F}_\ell) \neq 0$, $v_H: U_H/G \hookrightarrow X/G$ is the inclusion. Let $T = \bigcup_{H \in \mathcal{H}} V_H \subset V_G$.*

(a) *We have*

$$\overline{\text{Supp}}_G(H_G^*(X, j_!\mathbb{F}_{\ell U})) = T.$$

(b) *If $T \neq V_{\{0\}}$, then*

$$\overline{\text{Supp}}_G(H_G^{2*}(X, j_!\mathbb{F}_{\ell U})) = T.$$

(c) *For any $e \in S(\beta\check{G})$ such that $V_G(e) \supset T$, the restriction map $H_G^*(X, \mathbb{F}_\ell) \rightarrow H_G^*(Y, \mathbb{F}_\ell)$ induces an isomorphism*

$$H_G^*(X, \mathbb{F}_\ell)[e^{-1}] \xrightarrow{\sim} H_G^*(Y, \mathbb{F}_\ell)[e^{-1}]$$

of graded $H^(G, \mathbb{F}_\ell)[e^{-1}]$ -algebras.*

Note that \mathcal{H} is a subset of the set \mathcal{H}' of subgroups H of G such that $U_H \neq \emptyset$. For any maximal element H of \mathcal{H}' , $U_H = U^H$ is closed in U . Therefore, if $Y = \emptyset$, then \mathcal{H} and \mathcal{H}' have the same maximal elements, so that $T = \bigcup_{H \in \mathcal{H}'} V_H$.

Proof. By the long exact sequence of equivariant cohomology

$$(7.2.1) \quad \cdots \rightarrow H_G^*(X, j_!\mathbb{F}_{\ell U}) \rightarrow H_G^*(X) \rightarrow H_G^*(Y) \rightarrow \cdots,$$

(c) is equivalent to saying

$$H_G^*(X, j_!\mathbb{F}_{\ell U})[e^{-1}] = 0,$$

which is a consequence of (a).

For every subgroup H of G , we denote by $j_H: U_H \hookrightarrow X$ the immersion. We will show the following.

(a')

$$\text{Supp}_G(H_G^*(X, j_{H!}(\mathbb{F}_\ell)_{U_H})) = \begin{cases} V_H & \text{if } H \in \mathcal{H}, \\ \emptyset & \text{if } H \notin \mathcal{H}. \end{cases}$$

(b') For $H \in \mathcal{H}$ satisfying $H \neq \{0\}$, we have

$$\text{Supp}_G(H_G^{2*}(X, j_{H!}(\mathbb{F}_\ell)_{U_H})) = V_H.$$

Let us first prove that (a') and (b') imply (a) and (b). Note that $j_!(\mathbb{F}_{\ell U})$ is a successive extension of $j_{H!}(\mathbb{F}_\ell)_{U_H}$, H running through subgroups H of G . Thus (a') implies

$$(7.2.2) \quad \text{Supp}_G(H_G^*(X, j_!\mathbb{F}_{\ell U})) \subset \bigcup_H \text{Supp}_G(H_G^*(X, j_{H!}\mathbb{F}_\ell)) = T.$$

For any H , we denote by \mathfrak{p}_H the generic point of V_H , considered as a prime ideal of $S(\beta\check{G})$. The generic points of T are \mathfrak{p}_H , H running through maximal elements of \mathcal{H} . For any such H , (a') implies

$$H_G^*(X, j_{H!}\mathbb{F}_\ell)_{\mathfrak{p}_H} = 0$$

for all $H' \neq H$. Thus

$$H_G^*(X, j_!\mathbb{F}_{\ell U})_{\mathfrak{p}_H} \simeq H_G^*(X, j_{H!}\mathbb{F}_\ell)_{\mathfrak{p}_H} \neq 0.$$

In other words, $\text{Supp}_G(H_G^*(X, j_!\mathbb{F}_{\ell U}))$ contains all generic points of T . Combing this with (7.2.2), we obtain (a). If $T \neq V_{\{0\}}$, the origin of V_G is not a generic point of T . Then, as above, one deduces from (b') that $\text{Supp}_G(H_G^{2*}(X, j_!\mathbb{F}_{\ell U}))$ contains all generic points of T , which proves (b).

To show (a') and (b'), choose a subgroup H' of G such that $G = H \oplus H'$ and consider the Cartesian square

$$\begin{array}{ccc} U_H & \xrightarrow{u} & X^H \\ \downarrow f' & & \downarrow f \\ U_H/H' & \xrightarrow{v} & X^H/H' \end{array}$$

We have $H^*(G) \simeq H^*(H) \otimes_{\mathbb{F}_\ell} H^*(H')$ and isomorphisms of $H^*(G)$ -modules

$$(7.2.3) \quad H_G^*(X, j_{H!}\mathbb{F}_\ell) = H_G^*(X^H, u_!\mathbb{F}_\ell) \simeq H^*(H) \otimes_{\mathbb{F}_\ell} H_{H'}^*(X^H, u_!\mathbb{F}_\ell).$$

Here $\mathbb{F}_\ell = (\mathbb{F}_\ell)_{U_H}$. Since f' is a Galois étale cover of group H' , we have

$$(7.2.4) \quad R\Gamma_{H'}(X^H, u_!\mathbb{F}_\ell) \simeq R\Gamma_{H'}(X^H/H', f_*u_!\mathbb{F}_\ell) \simeq R\Gamma(X^H/H', R\Gamma(H', v_!f'_*\mathbb{F}_\ell)) \simeq R\Gamma(X^H/H', v_!\mathbb{F}_\ell).$$

As $\text{cd}_\ell(X^H/H')$ is finite (6.4), it follows that $H_{H'}^*(X^H, u_!\mathbb{F}_\ell)$ is of bounded degree. Thus, for every $y \in H'^\vee$,

$$H_{H'}^*(X^H, u_!\mathbb{F}_\ell)[(\beta y)^{-1}] = 0.$$

It follows that

$$\text{Supp}_{H'}(H_{H'}^*(X^H, u_!\mathbb{F}_\ell)) = \begin{cases} V_{\{0\}} & \text{if } H \in \mathcal{H}, \\ \emptyset & \text{if } H \notin \mathcal{H}, \end{cases}$$

which implies (a') by 7.3 (a) below applied to the projection $V_G \rightarrow V_{H'}$. Now let H be a nonzero element of \mathcal{H} . For nonzero elements $x \in \check{H}$ and $\alpha \in H_{H'}^i(X^H, u_!\mathbb{F}_\ell)$, $1 \otimes \alpha$ and $x \otimes \alpha$ in the right-hand side of (7.2.3) generate free sub- $S(\beta\check{G})$ -modules of rank 1 of $H_G^{2*+i}(X, j_{H!}\mathbb{F}_\ell)$ and $H_G^{2*+i+1}(X, j_{H!}\mathbb{F}_\ell)$, respectively. Thus

$$\text{Supp}_H(H_G^{2*}(X, j_{H!}\mathbb{F}_\ell)) = V_H.$$

Since $\text{Supp}_G(H_G^{2*}(X, j_{H!}\mathbb{F}_\ell)) \subset V_H$ by (a'), this implies (b') by 7.3 (b) applied to the projection $V_G \rightarrow V_H$. \square

Lemma 7.3. *Let $f: Y \rightarrow Z$ be a morphism of schemes.*

- (a) *Assume f is flat. Then for any quasi-coherent sheaf \mathcal{G} on Z , $\text{Supp}(f^*\mathcal{G}) = f^{-1}(\text{Supp}(\mathcal{G}))$.*
- (b) *Assume f is affine. Let \mathcal{F} be a quasi-coherent sheaf \mathcal{F} on Y of support contained in a subscheme Y_0 of Y such that $f|_{Y_0}: Y_0 \rightarrow Z$ is universally closed. Then $\text{Supp}(f_*\mathcal{F}) = f(\text{Supp}(\mathcal{F}))$.*

Proof. (a) For any point y of Y , since $\mathcal{O}_{Y,y}$ is faithfully flat over $\mathcal{O}_{Z,f(y)}$,

$$x \in \text{Supp}(f^*\mathcal{G}) \iff \mathcal{G}_{f(y)} \otimes_{\mathcal{O}_{Z,f(y)}} \mathcal{O}_{Y,y} = (f^*\mathcal{G})_y \neq 0 \iff \mathcal{G}_{f(y)} \neq 0 \iff f(y) \in \text{Supp}(\mathcal{G}).$$

(b) We may assume $Y = \text{Spec}(A)$, $Z = \text{Spec}(B)$. Let $\mathfrak{q} \in \text{Spec}(B)$. Then $\mathfrak{q} \in \text{Supp}(f_*\mathcal{F})$, i.e. $\mathcal{F} \otimes_B B_{\mathfrak{q}} \neq 0$, if and only if $(\mathcal{F} \otimes_B B_{\mathfrak{q}})_{\mathfrak{p}} \neq 0$ for some maximal ideal \mathfrak{p} of $A \otimes_B B_{\mathfrak{q}}$. By assumption, for any such \mathfrak{p} , $(f \otimes_B B_{\mathfrak{q}})(\mathfrak{p})$ is the closed point \mathfrak{q} of $\text{Spec}(B_{\mathfrak{q}})$. Thus $\mathfrak{q} \in \text{Supp}(f_*\mathcal{F})$ if and only if $\mathcal{F}_{\mathfrak{p}} \neq 0$ for some $\mathfrak{p} \in f^{-1}(\mathfrak{q})$, i.e. $\mathfrak{q} \in f(\text{Supp}(\mathcal{F}))$. \square

Applying 7.2 (c) to $j: X - X^G \hookrightarrow X$, we obtain the following analogue of Quillen's localization theorem [Qui71, 4.2].

Corollary 7.4. *Let X be an algebraic space separated and of finite type over k , endowed with an action of an elementary abelian ℓ -group G of rank r . Let $e = \prod_{x \in \check{G} - \{0\}} \beta x \in H^{2(\ell^r - 1)}(G, \mathbb{F}_\ell)$. Then the restriction map $H_G^*(X, \mathbb{F}_\ell) \rightarrow H_G^*(X^G, \mathbb{F}_\ell)$ induces an isomorphism*

$$H_G^*(X, \mathbb{F}_\ell)[e^{-1}] \xrightarrow{\sim} H_G^*(X^G, \mathbb{F}_\ell)[e^{-1}]$$

of graded $H^(G, \mathbb{F}_\ell)[e^{-1}]$ -algebras.*

Remark 7.5. (a) In 7.2, if $T = V_{\{0\}}$, it may happen that $H_G^{2*}(X, j_{!}\mathbb{F}_{\ell U}) = 0$. In fact, if $G = \{1\}$, X is mod ℓ acyclic (6.2), and Y is the disjoint union of n rational points, then $R\Gamma(X, j_{!}\mathbb{F}_{\ell U}) = \mathbb{F}_\ell^{n-1}[-1]$.

(b) In the situation of 7.4, assume G has rank 1. Since $H_G^*(X, j_{!}\mathbb{F}_\ell)$ is of bounded degree, (7.2.1) implies that the map

$$\rho: H_G^*(X) \rightarrow H_G^* \otimes H^0(X^G),$$

defined by the restriction $H_G^*(X) \rightarrow H_G^*(X^G) = H_G^* \otimes H^*(X^G)$ (6.3.5) composed with the projection onto $H_G^* \otimes H^0(X^G)$, has the following property: there exists an integer N such that, for any element $z \in \text{Ker } \rho$ (resp. $z \in H_G^* \otimes H^0(X^G)$) of positive degree, $z^N = 0$ (resp. $z^N \in \text{Im } \rho$). In a future paper [IZ11], we will discuss a generalization of this fact, analogous to Quillen's theorem [Qui71, 6.2].

(c) In the situation of 7.2, assume $\ell \neq p$. Then (7.2.3) and (7.2.4) imply that $H_G^*(X, j_{H!}\mathbb{F}_\ell)$ is a finitely generated $S(\beta\check{G})$ -module, because $H^*(X^H/H', v_{!}\mathbb{F}_\ell)$ is a finite-dimensional vector space (cf. 2.17). It follows that $H_G^*(X, j_{!}\mathbb{F}_{\ell U})$ is a finitely generated $S(\beta\check{G})$ -module and

$$\overline{\text{Supp}}_G(H_G^*(X, j_{!}\mathbb{F}_{\ell U})) = \text{Supp}_G(H_G^*(X, j_{!}\mathbb{F}_{\ell U})).$$

In the future paper, we will prove a finiteness result for more general groups G , analogous to Quillen's finiteness theorem [Qui71, 2.1].

Acknowledgements

This paper grew out of questions of J.-P. Serre. We thank him heartily for invaluable comments and suggestions. We also thank A. A. Beilinson for kindly communicating to us Rouquier's note [Rou], G. Laumon for discussions on equivariant cohomology, M. Olsson for giving us a proof of 6.4, A. Tamagawa for pointing out an error in a previous formulation of 3.8, and T. Saito for suggestions on § 2. The second author is grateful to A. Abbes for an invitation to l'Université de Rennes 1, where part of this work was done. We warmly thank the referee for a very careful reading of the manuscript and many helpful remarks.

References

- [Art69] M. Artin. The implicit function theorem in algebraic geometry. In *Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*, pages 13–34. Oxford Univ. Press, London, 1969.
- [Ber86] P. Berthelot. Géométrie rigide et cohomologie des variétés algébriques de caractéristique p . *Mém. Soc. Math. France (2)*, 23:7–32, 1986.
- [Ber97] P. Berthelot. Finitude et pureté cohomologique en cohomologie rigide. *Invent. Math.*, 128(2):329–377, 1997. With an appendix in English by Aise Johan de Jong.
- [BO78] P. Berthelot and A. Ogus. *Notes on crystalline cohomology*. Princeton University Press, Princeton, N.J., 1978.
- [Bor55] A. Borel. Nouvelle démonstration d'un théorème de P. A. Smith. *Comment. Math. Helv.*, 29:27–39, 1955.
- [Bre72] G. E. Bredon. *Introduction to compact transformation groups*, volume 46 of *Pure and Applied Mathematics*. Academic Press, New York, 1972.
- [CE56] H. Cartan and S. Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.
- [Con07] B. Conrad. Deligne's notes on Nagata compactifications. *J. Ramanujan Math. Soc.*, 22(3):205–257, 2007.
- [Del74] P. Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, 44:5–77, 1974.
- [Del80] P. Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, 52:137–252, 1980.
- [dJ96] A. J. de Jong. Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.*, 83:51–93, 1996.
- [DL76] P. Deligne and G. Lusztig. Representations of reductive groups over finite fields. *Ann. of Math. (2)*, 103(1):103–161, 1976.

- [Fuj02] K. Fujiwara. Independence of l for intersection cohomology (after Gabber). In *Algebraic geometry 2000, Azumino (Hotaka)*, volume 36 of *Adv. Stud. Pure Math.*, pages 145–151. Math. Soc. Japan, Tokyo, 2002.
- [Ful93] W. Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993.
- [Gir71] J. Giraud. *Cohomologie non abélienne*, volume 179 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1971.
- [GKM98] M. Goresky, R. Kottwitz, and R. MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998.
- [Gro85] M. Gros. Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique. *Mém. Soc. Math. France (N.S.)*, 21:1–87, 1985.
- [Hsi75] W.-Y. Hsiang. *Cohomology theory of topological transformation groups*, volume 85 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, New York, 1975.
- [Ill71] L. Illusie. *Complexe cotangent et déformations. I*. Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin, 1971.
- [Ill79] L. Illusie. Complexe de De Rham-Witt et cohomologie cristalline. *Ann. Sci. École Norm. Sup. (4)*, 12(4):501–661, 1979.
- [Ill81] L. Illusie. Théorie de Brauer et caractéristique d’Euler-Poincaré (d’après P. Deligne). In *Caractéristique d’Euler-Poincaré*, volume 82 of *Astérisque*, pages 161–172. Soc. Math. France, Paris, 1981.
- [Ill06] L. Illusie. Miscellany on traces in ℓ -adic cohomology: a survey. *Japanese J. Math.*, 1(1):107–136, 2006.
- [IZ11] L. Illusie and W. Zheng. An algebraic analogue of Quillen’s theory of equivariant cohomology algebras. In preparation, 2011.
- [Knu71] D. Knutson. *Algebraic spaces*, volume 203 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1971.
- [Lau76] G. Laumon. Homologie étale. In *Séminaire de géométrie analytique (École Norm. Sup., Paris, 1974–75)*, volume 36–37 of *Astérisque*, pages 163–188. Soc. Math. France, Paris, 1976.
- [Lau81] G. Laumon. Comparaison de caractéristiques d’Euler-Poincaré en cohomologie l -adique. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(3):209–212, 1981.
- [LMB00] G. Laumon and L. Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2000.
- [LO08] Y. Laszlo and M. Olsson. The six operations for sheaves on Artin stacks. II. Adic coefficients. *Publ. Math. Inst. Hautes Études Sci.*, 107:169–210, 2008.

- [LS07] B. Le Stum. *Rigid cohomology*, volume 172 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2007.
- [Lüt93] W. Lütkebohmert. On compactification of schemes. *Manuscripta Math.*, 80(1):95–111, 1993.
- [Mor08] B. Morin. *Sur le topos Weil-étale d’un corps de nombres*. PhD thesis, Université Bordeaux 1, 2008.
- [Noo04] B. Noohi. Fundamental groups of algebraic stacks. *J. Inst. Math. Jussieu*, 3(1):69–103, 2004.
- [PR86] T. Petrie and J. D. Randall. Finite-order algebraic automorphisms of affine varieties. *Comment. Math. Helv.*, 61(2):203–221, 1986.
- [Qui71] D. Quillen. The spectrum of an equivariant cohomology ring. I. *Ann. of Math. (2)*, 94:549–572, 1971.
- [Ric94] J. Rickard. Finite group actions and étale cohomology. *Publ. Math. Inst. Hautes Études Sci.*, 80:81–94, 1994.
- [Rou] R. Rouquier. On permutation modules. Private note.
- [Ser98] J.-P. Serre. *Représentations linéaires des groupes finis*. Hermann, Paris, 5th edition, 1998.
- [Ser07] J.-P. Serre. Bounds for the orders of the finite subgroups of $G(k)$. In *Group representation theory*, pages 405–450. EPFL Press, Lausanne, 2007.
- [Ser09] J.-P. Serre. How to use finite fields for problems concerning infinite fields. In *Arithmetic, Geometry, Cryptography and Coding Theory*, volume 487 of *Contemp. Math.*, pages 183–193. Amer. Math. Soc., Providence, RI, 2009.
- [Ser10] J.-P. Serre. Le groupe de Cremona et ses sous-groupes finis. *Astérisque*, 332:75–100, 2010. Séminaire Bourbaki, Nov. 2008, No. 1000.
- [Smi38] P. A. Smith. Transformations of finite period. *Ann. of Math. (2)*, 39(1):127–164, 1938.
- [Sul71] D. Sullivan. Combinatorial invariants of analytic spaces. In *Proceedings of Liverpool Singularities—Symposium, I (1969/70)*, pages 165–168, Berlin, 1971. Springer.
- [Sym04] P. Symonds. Smith theory for algebraic varieties. *Algebr. Geom. Topol.*, 4:121–131, 2004.
- [The] The Stacks Project Authors. *Stacks Project*. http://math.columbia.edu/algebraic_geometry/stacks-git.
- [Tsu03] N. Tsuzuki. Cohomological descent of rigid cohomology for proper coverings. *Invent. Math.*, 151(1):101–133, 2003.

- [Ver73] J.-L. Verdier. Caractéristique d'Euler-Poincaré. *Bull. Soc. Math. France*, 101:441–445, 1973.
- [Vid04] I. Vidal. Théorie de Brauer et conducteur de Swan. *J. Algebraic Geom.*, 13(2):349–391, 2004.
- [Vid05] I. Vidal. Courbes nodales et ramification sauvage virtuelle. *Manuscripta Math.*, 118(1):43–70, 2005.
- [Zhe09] W. Zheng. Sur l'indépendance de l en cohomologie l -adique sur les corps locaux. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(2):291–334, 2009.
- [Zhe11] W. Zheng. Six operations on Deligne-Mumford stacks. [arXiv:1006.3810](https://arxiv.org/abs/1006.3810), 2011.
- [EGA II] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Publ. Math. Inst. Hautes Études Sci.*, 8:5–222, 1961.
- [EGA IV] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. *Publ. Math. Inst. Hautes Études Sci.*, 20:5–259, 24:5–231, 28:5–255, 32:5–361, 1964–1967.
- [SGA 1] *Revêtements étales et groupe fondamental*, volume 3 of *Documents Mathématiques (Paris)*. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–1961 (SGA 1), dirigé par A. Grothendieck, édition recomposée et annotée.
- [SGA 4] *Théorie des topos et cohomologie étale des schémas*, volume 269, 270, 305 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), dirigé par M. Artin, A. Grothendieck, et J.-L. Verdier.
- [SGA 4 $\frac{1}{2}$] P. Deligne. *Cohomologie étale*, volume 569 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 $\frac{1}{2}$, avec la collaboration de J.-F. Boutot, A. Grothendieck, L. Illusie et J.-L. Verdier.
- [SGA 5] *Cohomologie ℓ -adique et fonctions L* , volume 589 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie 1965–1966 (SGA 5), dirigé par A. Grothendieck.